

# On the Jarque—Bera normality test

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## Abstract

The Jarque-Bera normality test verify if the residues  $u_i = Y_i - A_0 - \sum_{j=1}^k A_j X_j^{(i)}$  of the regression hyper-plane  $Y = A_0 + \sum_{i=1}^k A_i X_i$  are normal random variables.

We can prove that if  $Y_i$  are normal then  $u_i$  are also normal random variables. If the Jarque-Bera test fails, we have to use a transformation  $Z_i = G(Y_i)$  such that  $Z_i$  are normal, we find the regression hyper-plane  $Z = A_0 + \sum_{i=1}^k A_i X_i$ , and from here we obtain the regression  $Y = G^{-1}\left(A_0 + \sum_{i=1}^k A_i X_i\right)$ .

For this we consider the case when we know the pdf, the cdf and the inverse of the cdf for the random variable  $Y$  (example: the exponential distribution), the case when we know only the first two elements (example: Erlang distribution) and the case when we know only the pdf (example: the gamma distribution).

We consider also the case when we do not know even an analytical formula for the pdf. In this case we will estimate the pdf using some known kernels (see section 2).

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## 1 Introduction

Consider  $n$  points in  $\mathbb{R}^{k+1}$ ,  $X^{(1)}, \dots, X^{(n)}$ , where  $X^{(i)} = (X_1^{(i)}, X_2^{(i)}, \dots, X_k^{(i)}, Y_i)$ .

The regression hyper-plane used in (Ciuiu 2007a) to classify patterns has the equation (Saporta 1990)

$$H : Y = A_0 + \sum_{i=1}^k A_i X_i \text{ such that} \quad (1)$$

$$\sum_{i=1}^n u_i^2 \text{ is minimum,} \quad (1')$$

where the residues  $u_i$  have the formula

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$$u_i = Y_i - A_0 - \sum_{j=1}^k A_j X_j^{(i)}. \quad (1'')$$

For the computation of  $A_i$  from (1) we have to solve the system (Saporta, 1990)

$$\sum_{j=0}^k \overline{X_i \cdot X_j} \cdot A_j = \overline{X_i \cdot Y}, \quad i = \overline{0, k}, \quad (2)$$

where  $\overline{X_0 \cdot X_i} = \overline{X_i}$  and  $\overline{X_0^2} = 1$ .

The obtained estimators of  $A_i$  using (2) and of the residues  $u_i$  have the following properties (Jula 2003, Voineagu et al 2007):

1. The estimators of  $A_i$  are linear.
2. The estimators of  $A_i$  are consistent.
3. The estimators of  $A_i$  are unbiased.
4. The estimators of  $A_i$  have the minimum variance.
5. The estimators of  $A_i$  have the maximum likelihood.
6. The estimators of  $u_i$  have the expectation 0 and the same variance (homo-scedasticity).
7. The estimators of  $u_i$  are normal.

In (Jula 2003) there is presented a test for the normality of  $u_i$ . First we compute the skewness:

$$S = \frac{\overline{u^3}}{\left(\overline{u^2}\right)^{\frac{3}{2}}}, \quad (3)$$

and the kurtosis:

$$K = \frac{\overline{u^4}}{\left(\overline{u^2}\right)^2}. \quad (4)$$

The Jarque—Bera statistics is

$$JB = n \left( \frac{S^2}{6} + \frac{(K - 3)^2}{24} \right). \quad (5)$$

For the normal distribution of  $u_i$  we have  $S = 0$ ,  $K = 3$  and  $JB = 0$ .  $JB$  has in fact the distribution  $\chi_2^2$  ( $\chi^2$  with two degrees of freedom), hence we accept the null hypothesis  $H_0 : S = 0$  and  $K = 3$  with the error  $\varepsilon$  if and only if  $JB < \chi_{2;\varepsilon}^2$ .

If the number of points is large enough we can use the central limit theorem to obtain the residues asymptotically normals. Even the eigenvalues from the *MCA* (Multiple Correspondence Analysis) are asymptotically normals (Saporta and Ben Ammou 1998). But if we have only 25 points these results cannot be applied, as we can see in section 3.

## 2 Making normal residues

In this section we consider the case when the Jarque—Bera test fails. Because when we solve the linear system 2 by the Cramér method we can notice that all the estimators are linear in  $Y_1, \dots, Y_n$ , we have the following proposition.

**Proposition 1** *If  $Y_i$  have a normal distribution then the residues  $u_i$  have a normal distribution.*

In (Jula 2003) there are presented two method to treat the case when the residues are not normal. The first method consist in the identification of the distribution of  $u_i$ , and to apply a nonlinear regression. The second one is to transform  $Y_i$  and/or  $X_j^{(i)}$  to eliminate the non-normality.

We will use the second method and the proposition 1: we transform  $Y_i$  to obtain normal variables. Suppose the pdf of  $Y_i$  is  $f$ , the cdf of  $Y_i$  is  $F$  and the inverse of the cdf is  $F^{-1}$ .

When we know analytical formulae for  $f$ ,  $F$  and  $F^{-1}$  we make the substitution

$$Z_i = \Phi^{-1} \circ F(Y_i), \quad (6)$$

we compute the regression hyper-plane (1) with  $Z$  instead of  $Y$ , and we obtain the regression

$$Y = F^{-1} \circ \Phi \left( A_0 + \sum_{i=1}^k A_i X_i \right). \quad (7)$$

The pdf of a normal random variable  $N(0, 1)$  is denoted by  $\phi(x)$  and the cdf is  $\Phi$ .

When we do not know an analytical formula for  $F^{-1}$  (as in the normal case  $F = \Phi$ ) we estimate  $F^{-1}(x)$  by the Monte Carlo method as follows.

First we generate 100 random variables uniform on  $[0, 1]$ . For each of the generated values  $y$  we define  $\Psi(y) = F(-\ln y)$  if  $F(0) < x$ , respectively  $\Psi(y) = F(\ln y)$  if  $F(0) > x$ .

In both cases a primitive of  $F(y) - x$  is  $\tilde{\Psi}(y) = -x \cdot y - \int_y^1 \Psi(t) dt$ . From these values we take  $F^{-1}(x) = \tilde{y}$  such that

$$-\tilde{\Psi}(y) = x \cdot y + \int_y^1 \Psi(t) dt \quad (8)$$

has its minimum in  $\tilde{y}$  if  $F(0) < x$ , respectively its maximum if  $F(0) > x$ . Of course, if  $F(0) = x$  we have  $F^{-1}(y) = 0$ . The integral from (8) is computed by the Monte Carlo method: we generate 100 uniform random variables  $Z_i$  in  $[y, 1]$  and we use the estimation

$$\int_y^1 \Psi(t) dt = \frac{\sum_{i=1}^{100} (1 - y) \Psi(Z_i)}{100}. \quad (8')$$

When we know only an analytical formula for the pdf  $f$  we generate first 100 random variables  $U_i$  on  $[0, 1]$  and we estimate

$$F(x) = \int_{-\infty}^x f(t)dt = \int_0^1 \frac{f(x + \ln u)}{u} du = \sum_{i=1}^{100} \frac{f(x + \ln U_i)}{100 * U_i}. \quad (9)$$

For the estimation of  $\Phi^{-1}$  in (6) we use (8) and for the estimation of  $\Phi$  in (7) we use (9).

The estimation (9) is used to estimate the cdf if the support of the random variable is  $\mathbb{R}$ . If this support is  $[0, \infty)$  we generate also 100 uniform random variables on  $[0, 1]$ , but we use the estimation

$$F(x) = \int_0^x f(t)dt = \int_0^1 x \cdot f(x \cdot u) du = \frac{\sum_{i=1}^{100} x \cdot f(x \cdot U_i)}{100}. \quad (9')$$

This is the case of the Gamma distribution. In fact we do not know the pdf because in

$$f(x) = \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}}{\Gamma(\alpha) \cdot \beta^\alpha}. \quad (10)$$

we do not know  $\Gamma(\alpha)$ . But we use the estimation

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-2} \cdot e^{-t} dt = \frac{\sum_{i=1}^{1000} Z_i^{\alpha-1}}{1000}, \quad (10')$$

where  $Z_i$  are 1000 random variables with the distribution  $\exp(1)$ .

In fact in the case of the Gamma distribution we estimate only once  $\Gamma(\alpha)$ : we use  $\Gamma(\alpha) \cdot f(x)$  instead of  $f(x)$ ,  $\Gamma(\alpha) \cdot F(x)$  instead of  $F(x)$  and  $F^{-1}(\Gamma(\alpha) \cdot x)$  instead of  $F^{-1}(x)$ .

When we do not know even the pdf  $f$  we can estimate this pdf using some kernels (Văduva 1968, Văduva and Pascu 2003). If we have the  $n$ -size sample  $X_1, \dots, X_n$  we estimate the pdf  $f$  in  $x$  by

$$f(x) = \frac{1}{n \cdot b_n} \cdot \sum_{i=1}^n K\left(\frac{x - X_i}{b_n}\right), \quad (11)$$

where  $K$  is the kernel and  $b_n$  is the window bandwidth.

There are some kernel functions  $K = K_j$  used in literature:

$$\left\{ \begin{array}{l} K_0(x) = \chi_{-\frac{1}{2}, \frac{1}{2}}(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{if } |x| > \frac{1}{2} \end{cases} \\ K_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ K_2(x) = \frac{3}{4\sqrt{5}} \left(1 - \frac{x^2}{5}\right) \chi_{-\sqrt{5}, \sqrt{5}}(x) \\ K_3(x) = (1 - |x|) \chi_{-1, 1}(x) \\ K_4(x) = \frac{4}{3} (1 - x^2) \chi_{-1, 1}(x) \\ K_5(x) = \frac{15}{16} (1 - x^2)^2 \chi_{-1, 1}(x) \end{array} \right. , \quad (12)$$

where  $K_0$  is the rectangular kernel,  $K_1$  is the Gaussian kernel,  $K_2$  is the Epanechnikoff

kernel,  $K_3$  is the triangular kernel,  $K_4$  is the Bartlett—Priestley—Epanechnikoff kernel and  $K_5$  is the biquadratic kernel.

The Epanechnikoff kernel  $K_2$  minimizes the MISE (Min Integrated Square Error)

$$MISE(\hat{h}) = \int_{-\infty}^{\infty} MSE_x(\hat{h}) dx, \quad (13)$$

where  $MSE_x(\hat{h})$  is the mean square error (Văduva 1968, Văduva and Pascu 2003):

$$MSE_x(\hat{h}) = E \left\{ \left( \hat{h}(x) - h(x) \right)^2 \right\}. \quad (13')$$

The window bandwidth must be choosed such that  $\lim_{n \rightarrow \infty} b_n = 0$  and  $\lim_{n \rightarrow \infty} n \cdot b_n = \infty$  (Văduva 1968, Văduva and Pascu 2003). In our C++ program we take  $b_n = \frac{1}{\sqrt{n}}$  and we obtain

$$f(x) = \frac{1}{\sqrt{n}} \cdot \sum_{i=1}^n K(\sqrt{n}(x - X_i)). \quad (11')$$

Sometimes we have a shortcut for obtaining  $Z_i$  in (6). For instance, if the distribution of  $Y$  is log-normal we can do the substitution

$$Z_i = \ln Y_i, \quad (6')$$

and (7) becomes

$$Y = \exp \left( A_0 + \sum_{i=1}^k A_i X_i \right). \quad (7')$$

The integrals that appear in the estimation of the cdf (and its inverse) are computed in the classical way by the Monte Carlo method, as the Neyman or the Neyman—Bayes estimator of the parameter is computed by an integral formula (Ciuiu 2007b). The centil of the  $\chi^2$  distribution with two degrees of freedom is computed by the formula  $\chi_{2;\varepsilon}^2 = -2 \ln \varepsilon$ , because the  $\chi_2^2$  distribution is identical to the  $\exp(\frac{1}{2})$  distribution (Ciuiu 2005, Ciuiu 2006).

### 3 Application

Consider the following consumer behavior model with 25 customers, where  $X_1$  represents the advertisement,  $X_2$  represents the prices and  $Y$  represents the sales (Jula 2003):

$X_1$	3	2	0.8	2.5	2	1.4	2.5	2.5	3	1.4	1	1.2	1.6
$X_2$	1.3	2.8	1.5	0.2	1.8	4	1.8	2	0.5	2.8	3.2	2.5	1.3
$Y$	2	0.5	1.5	3	1	0	2.1	1.8	3	0.7	0.5	1	1.4
$X_1$	1.8	1	2.8	3.5	2.6	2.4	3.4	1.6	1.9	3.5	1.6	3	
$X_2$	2.2	3.5	1.1	0	0.2	2	1.2	3	3	0.6	3.2	0.3	
$Y$	1.2	0.8	2.3	3.5	3.8	1.8	2.6	0.8	1.2	4.2	0.8	2.5	

The regression hyper-plane is  $Y = 1.98362 + 0.44053X_1 - 0.63867X_2$ , the skewness is  $S = 0.57887$  and the kurtosis is  $K = 3.65375$ . Because the Jarque Bera statistics is  $JB =$

1.8414 and the  $\chi_2$  cenil of the error 0.05 is  $\chi_{2,0.05}^2 = 5.99146$ , we have  $1.84140 < 5.99146$ , hence we accept that the residues are normal. these results are also obtained in (Jula 2003).

The goal of a regression is to estimate the value of  $Y$  if we know the values of  $X_1, \dots, X_k$ . For instance if  $X_1 = 5.25$  and  $X_2 = 1.33$  we obtain  $Y = 3.44695$ .

If the Jarque—Bera test fails we can conclude that neither  $Y$  is normal. In this case we denote by  $U = F(Y)$  the obtained uniform random variable and by  $Z = \Phi^{-1}(U)$ .

Suppose that  $Y$  has an exponential distribution and the following values obtained by simulation: (2.67216, 6.10767, 0.16366, 9.09858, 0.44079, 3.37261, 0.75242, 1.99623, 1.85102, 0.38951, 0.45893, 0.09252, 0.79938, 1.81434, 0.19946, 0.88753, 3.34587, 1.17205, 6.8799, 0.8957, 11.29056, 4.1833, 0.336, 0.89712, 5.29434). The regression hyper-plane is  $Y = -0.14080 + 0.90549X_1 + 0.43511X_2$ , the skewness is  $S = 1.62411$  and the kurtosis is  $K = 4.87035$ . Because the Jarque Bera statistics is  $JB = 14.63450$  we have  $14.6345 > 5.99146$ , hence we reject that the residues are normal. The uniform variables  $U$  are: (0.63998, 0.90319, 0.06065, 0.96915, 0.15508, 0.72456, 0.24998, 0.53382, 0.50721, 0.13836, 0.16092, 0.03475, 0.26333, 0.50025, 0.07342, 0.28774, 0.72173, 0.36115, 0.92794, 0.28996, 0.98665, 0.79797, 0.12055, 0.29035, 0.86789). The normal variables  $Z$  are: (0.38946, 4.25514,  $-1.46483$ , 1.85666,  $-0.83021$ , 1.14454,  $-0.89828$ , 1.34244, 0,  $-1.02412$ ,  $-1.0231$ ,  $-1.86904$ ,  $-0.47772$ , 0,  $-1.85315$ ,  $-0.27203$ , 0.79859,  $-0.38663$ , 2.38615,  $-0.23645$ , 0.3253, 2.73142,  $-1.46893$ ,  $-0.60055$ , 0.67497). We obtain the regression  $Y = F^{-1} \circ \Phi(-3.30458 + 1.07879X_1 + 0.60564X_2)$ , where  $F(x) = 1 - e^{-\lambda x}$  (the exponential cdf) and  $\Phi$  is the cdf of the standard normal distribution. The new skewness is  $S = 0.85063$ , the new kurtosis is  $K = 3.55469$  and the new Jarque Bera statistics is  $JB = 3.33535$ . Because  $3.33535 < 5.99146$  we accept now that the new residues are normal. If  $X_1 = 2.5$  and  $X_2 = 1.67$  we obtain  $Y = 3.07021$ .

If  $Y$  has Erlang distribution with the values (2.96613, 6.48113, 4.08712, 3.89862, 9.03531, 4.17294, 4.20856, 0.4176, 2.77439, 0.86518, 3.54838, 1.27318, 4.09422, 0.66227, 1.37381, 2.57038, 2.301, 4.47228, 1.0578, 2.63542, 4.46888, 2.00138, 2.14516, 2.69531, 2.65117) we obtain the regression hyper-plane  $Y = 2.87639 + 1.15495X_1 + 0.70305X_2$ , the skewness  $S = 1.57764$  and the kurtosis  $K = 6.80432$ . Because the Jarque Bera statistics is  $JB = 25.4465$  we have  $25.4465 > 5.99146$ , hence we reject that the residues are normal. The uniform variables  $U$  are: (0.76045, 0.59577, 0.35627, 0.42475, 0.57767, 0.04391, 0.82315, 0.86564, 0.73634, 0.81381, 0.43409, 0.5787, 0.76926, 0.04632, 0.9632, 0.99731, 0.75557, 0.5703, 0.7628, 0.19752, 0.80786, 0.94658, 0.5907, 0.63524, 0.50501). The normal variables  $Z$  are: (0.82679, 0.5443,  $-0.35593$ ,  $-0.07781$ , 0.63623,  $-1.09636$ , 0.64865, 1.75972, 0.80662, 0.11085,  $-0.19447$ , 3.34705, 0.83808,  $-2.36028$ , 1.7295, 2.19251, 1.66568, 3.68058, 0.75272,  $-0.57019$ , 0.4333, 3.68665, 0.43424, 0.28145, 0.04798). If we denote by  $F$  the cdf of the Erlang distribution, we obtain the regression  $Y = F^{-1} \circ \Phi(1.09421 - 0.00690X_1 - 0.15684X_2)$ , the new skewness  $S = 0.45896$  and the new kurtosis  $K = 3.3914$ . Because the new Jarque Bera statistics is  $JB = 1.03726$  we have  $1.03726 < 5.99146$ , hence we accept that the residues are normal. If  $X_1 = 7.5$  and  $X_2 = 2.25$  we obtain using the above regression  $Y = 4.24444$ .

If  $Y$  has a Gamma distribution with the values (0.59245, 0.12004, 0.41625, 0.43813, 6.84824, 0.74769, 0.25507, 1.29952, 0.87027, 9.33513, 10.64632, 0.25897, 6.29302, 6.57701, 13.60256, 18.23679, 1.15156, 0.57335, 0.11935, 6.59902, 8.98237, 9.64063, 0.11099, 0.83072, 0.59731) we obtain the regression hyper-plane  $Y = 3.11599 - 0.47429X_1 + 1.14901X_2$ , the skewness  $S = 1.20115$  and the kurtosis  $K = 4.81751$ . Because the Jarque Bera statistics is  $JB = 9.45247$  we have  $9.45247 > 5.99146$ , hence we reject that the residues are normal. The uniform variables  $U$  are: (0.2, 0.07184, 0.16171, 0.17124, 0.70923, 0.22844, 0.12229,

0.33659, 0.24844, 0.79108, 0.79108, 0.11887, 0.78608, 0.75107, 0.791, 0.8, 0.35172, 0.19326, 0.07132, 0.75337, 0.79108, 0.79108, 0.06911, 0.26843, 0.20799). The normal variables  $Z$  are:  $(-1.34396, -1.16078, -0.94806, -0.8641, 0.28339, -0.77506, -1.37933, -0.87488, 0.13259, 1.02073, 1.04595, -1.08383, 1.99092, 0.79554, 0.51645, 0.33609, -0.12237, -0.67827, -2.80078, 0.09764, 1.09563, 0.27315, -1.78141, -0.79649)$ . If we denote by  $F$  the cdf of the Gamma distribution we obtain the regression  $Y = F^{-1} \circ \Phi(1.22080 - 0.58973X_1 - 0.14010X_2)$ , the new skewness  $S = -0.25517$  and the new kurtosis  $K = 2.43605$ . Because the new Jarque Bera statistics is  $JB = 0.60259$  we have  $0.60259 < 5.99146$ , hence we accept that the residues are normal. If  $X_1 = 2.25$  and  $X_2 = 3.14$  we obtain using the above regression  $Y = 0.85029$ .

If the pdf of the random variable  $Y$  is unknown we use the same data as in the case of the exponential distribution, hence we reject first the normality. For the rectangular kernel we obtain the uniform variables  $(0.74256, 1, 0.08428, 0.07175, 0.172, 0.89306, 0.31327, 0.4615, 0.56391, 0.17807, 0.2052, 0.02922, 0.44244, 0.65745, 0.08622, 0.4619, 0.67617, 0.50883, 1, 0.42426, 0.58903, 1, 0.12203, 0.44248, 1)$  and the normal variables  $(0.64697, 6.38671, -1.38233, -1.55841, -1.05078, 4.59092, -0.57953, -0.09035, 0.16386, -0.97852, -0.9208, -3.65163, -0.15038, 0.53152, -1.66153, -0.1049, 0.49897, 0.00335, 4.36347, -0.20497, 0.25379, 2.28768, -1.26589, -0.15907, 4.87802)$ . The regression becomes  $Y = F^{-1} \circ \Phi(-5.52184 + 1.74946X_1 + 1.18307X_2)$  with the new skewness  $S = 1.01127$ , the new kurtosis  $K = 3.40948$  and the new Jarque Bera statistics  $JB = 4.43579$ . Therefore we accept that the residues are normal because  $4.43579 < 5.99146$ . If we take  $X_1 = 1.25$  and  $X_2 = 3$  we obtain using the above regression  $Y = 1.3844$ .

For the Gaussian kernel we obtain the uniform variables  $(0.73093, 0.89068, 0.07445, 0.11027, 0.24439, 0.55138, 0.3344, 0.64391, 0.49156, 0.15477, 0.21829, 0.06058, 0.36889, 0.53458, 0.11761, 0.43847, 0.59584, 0.46749, 0.20761, 0.39841, 0.07033, 1, 0.16865, 0.43483, 0.69935)$  and the normal variables  $(0.90005, 1.46789, -1.93828, -1.24249, -0.73297, 0.11938, -0.46527, 0.35026, -0.02663, -1.15884, -0.90466, -2.29383, -0.30879, 0.14237, -0.996, -0.1637, 0.30081, -0.0756, -1.03623, -0.24585, -2.15782, 2.92437, -0.88029, -0.15409, 0.5808)$ . The regression becomes  $Y = F^{-1} \circ \Phi(-3.57654 + 1.05118X_1 + 0.53597X_2)$  with the new skewness  $S = 0.72574$ , the new kurtosis  $K = 4.53733$  and the new Jarque Bera statistics  $JB = 4.65644$ . Therefore we accept that the residues are normal because  $4.65644 < 5.99146$ . If we take  $X_1 = 4.5$  and  $X_2 = 2.25$  we obtain using the above regression  $Y = 3.00951$ .

For the Epanechnikoff kernel we obtain the uniform variables  $(0.689, 0.38222, 0.08727, 0.71996, 0.21662, 0.80481, 0.39366, 0.54299, 0.55838, 0.16742, 0.21597, 0.05918, 0.3584, 0.61796, 0.10216, 0.40666, 0.72927, 0.48629, 0.69491, 0.42013, 0.07246, 0.55466, 0.1875, 0.39567, 0.59211)$  and the normal variables  $(0.57115, -0.34765, -1.60459, 0.6327, -0.81905, 1.03572, -0.2772, 0.10761, 0.14342, -1.04711, -0.76504, -1.53142, -0.43606, 0.26439, -1.31213, -0.24027, 0.85928, -0.03952, 0.6869, -0.21434, -2.19226, 0.12365, -0.8131, -0.23626, 0.21729)$ . The regression becomes  $Y = F^{-1} \circ \Phi(-2.02022 + 0.67981X_1 + 0.14265X_2)$  with the new skewness  $S = -0.27616$ , the new kurtosis  $K = 3.54591$  and the new Jarque Bera statistics  $JB = 0.6282$ . Therefore we accept that the residues are normal because  $0.6282 < 5.99146$ . If we take  $X_1 = 2$  and  $X_2 = 3.75$  we obtain using the above regression  $Y = 1.09615$ .

For the triangular kernel we obtain the uniform variables  $(0.55864, 1, 0.08007, 0.22144, 0.22309, 0.64824, 0.31701, 0.56918, 0.59285, 0.21256, 0.22223, 0.02085, 0.33645, 0.483, 0.08317, 0.40398, 0.59495, 0.44167, 0.23361, 0.40534, 1, 0.87282, 0.12986, 0.47971, 1)$  and the normal variables  $(0.1888, 1.70783, -1.46731, -0.84107, -0.84048, 0.43218, -0.47812,$

0.20582, 0.18546, -0.80444, -0.92568, -2.47714, -0.45085, -0.04064, -1.20328, -0.26797, 0.28383, -0.16019, -0.73053, -0.21772, 2.00289, 2.86874, -1.04085, -0.04609, 1.99734). The regression becomes  $Y = F^{-1} \circ \Phi(-3.24280 + 0.96121X_1 + 0.58793X_2)$  with the new skewness  $S = 0.79678$ , the new kurtosis  $K = 3.32667$  and the new Jarque Bera statistics  $JB = 2.75639$ . Therefore we accept that the residues are normal because  $2.75639 < 5.99146$ . If we take  $X_1 = 1.5$  and  $X_2 = 1.75$  we obtain using the above regression  $Y = 0.44912$ .

For the Bartlett—Priestley—Epanechnikoff kernel we obtain the uniform variables (1, 0.33857, 0.13812, 0.12596, 0.44539, 0.73052, 0.5933, 1, 1, 0.25732, 0.34066, 0.07253, 0.68407, 0.90392, 0.13608, 0.81143, 1, 0.84355, 0.48027, 0.90592, 0.16949, 0.89472, 0.27181, 0.83611, 0.22518) and the normal variables (3.17017, -0.47278, -1.14953, -1.36666, -0.12689, 0.62876, 0.27496, 2.6231, 3.25374, -0.68632, -0.36865, -1.60882, 0.39331, 4.46687, -1.08116, 1.8969, 3.03824, 0.82114, -0.05247, 2.68604, -1.0866, 2.09657, -0.62238, 1.11741, -0.77158). The regression becomes  $Y = F^{-1} \circ \Phi(-3.51351 + 1.52175X_1 + 0.49427X_2)$  with the new skewness  $S = 0.57828$ , the new kurtosis  $K = 3.66344$  and the new Jarque Bera statistics  $JB = 1.85185$ . Therefore we accept that the residues are normal because  $1.85185 < 5.99146$ . If we take  $X_1 = 1$  and  $X_2 = 3.14$  we obtain using the above regression  $Y = 0.47805$ .

For the biquadratic kernel we obtain the uniform variables (0.46534, 0.45856, 0.05495, 0.84543, 0.15733, 0.46435, 0.2533, 0.50861, 0.58318, 0.20579, 0.2491, 0.03261, 0.30475, 0.64295, 0.06771, 0.4374, 0.57995, 0.52029, 1, 0.45258, 0.0677, 0.23793, 0.14287, 0.40738, 0.14171) and the normal variables (-0.07511, -0.09869, -1.70286, 1.79589, -1.28617, -0.09582, -0.70913, 0.01464, 0.28983, -0.84288, -0.68003, -3.5665, -0.4849, 0.41251, -2.10384, -0.17375, 0.18491, 0.03588, 1.66149, -0.12889, -1.55505, -0.78461, -0.95044, -0.23288, -1.0972). The regression becomes  $Y = F^{-1} \circ \Phi(-1.97926 + 0.66020X_1 + 0.03603X_2)$  with the new skewness  $S = -0.02359$ , the new kurtosis  $K = 3.8186$  and the new Jarque Bera statistics  $JB = 0.70035$ . Therefore we accept that the residues are normal because  $0.70035 < 5.99146$ . If we take  $X_1 = 2.33$  and  $X_2 = 1.67$  we obtain using the above regression  $Y = 1.1542$ .

If the random variable  $Y$  is log-normal with the values (7.38906, 1.64872, 4.48169, 20.08554, 2.71828, 1, 8.16617, 6.04965, 20.08554, 2.01375, 1.64872, 2.71828, 4.0552, 3.32012, 2.22554, 9.97418, 33.11545, 44.70118, 6.04965, 13.46374, 2.22554, 3.32012, 66.68633, 2.22554, 12.18249) we obtain the regression hyper-plane  $Y = 8.02697 + 6.34842X_1 - 5.69431X_2$ , the skewness  $S = 2.06867$  and the kurtosis  $K = 8.03926$ . Because the Jarque Bera statistics is  $JB = 44.28315$  we have  $44.28315 > 5.99146$ , hence we reject that the residues are normal. The normal variables  $Z$  are the same as in the normal case (the log-normal values were obtained by the exponential). Therefore we obtain the same skewness, kurtosis and Jarque—Bera statistics as in the normal case, but the new regression is  $Y = \exp(1.98362 + 0.44053X_1 - 0.63867X_2)$ . If  $X_1 = 3.5$  and  $X_2 = 1.25$  we obtain using the above regression  $Y = 15.28882$ .

## 4 Conclusions

When the Jarque—Bera test fails we have to transform the random variable  $Y$  in a normal one using (6) and proposition 1. If we want to compute a new value of  $Y$  when we know new values for the explanatory variables we do the inverse transformation.

If we know analytical formulae for the pdf  $f$ , the cdf  $F$  and the inverse of the cdf  $F^{-1}$  we use the formula (6). If some of the above elements are not known we use the estimators from the second section.

Instead of the estimators obtained in this paper by the Monte Carlo method we can

use numerical methods to compute the cdf  $F$  (numerical integrals) and the inverse of  $F$  (numerical methods to solve nonlinear equations). An open problem is to create an analogue  $C++$  program using numerical methods.

The obtained normal variables  $Z$  are standard ones. Another open problem is to use some other normal random variables (with other expectation and other variance).

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