Economic Computation and Economic Cybernetics Studies and Research, Issue 1/2016, Vol. 50

Associate Professor Xinyu WU, PhD
E-mail: xywu@hotmail.com
School of Finance, Anhui University of Finance and Economics, China
Professor Hailin ZHOU, PhD
E-mail: hailin_zhou@126.com
School of Finance, Anhui University of Finance and Economics, China

GARCH DIFFUSION MODEL, iVIX, AND VOLATILITY RISK PREMIUM

Abstract. *This paper investigates the volatility risk premium in the non-affine GARCH diffusion model of stochastic volatility using the Chinese Volatility Index (iVIX). Firstly, we derive the corresponding implied iVIX formula under the GARCH diffusion model. Then, using joint data on the Shanghai 50ETF and iVIX index, we develop an efficient importance sampling (EIS)-based joint maximum likelihood (ML) estimation method for the objective and risk-neutral parameters of the GARCH diffusion model. Furthermore, a particle filter-based estimation method is developed for extracting the latent volatility. Finally, we apply our proposed approach to the actual data on the Shanghai 50ETF and iVIX index. Empirical results show that the volatility risk is priced by the market, and the volatility risk premium is negative, implying that investors act risk averse in the Shanghai stock market.*

Keywords: *GARCH diffusion model, iVIX, volatility risk premium, efficient importance sampling, particle filter.*

JEL Classification: C13, C32, C58, G13

1. Introduction

Finance literature has put much effort on studying the volatility risk premium. According to modern asset pricing theory, the value of any contingent claim can be computed as the conditional expectation under the risk-neutral measure of the discounted future cash lows. Thus, the valuation of any contingent claim, like a European option, involves a change of the measure, from the objective or real-world measure to the risk-neutral measure. However, the characterization of the risk-neutral measure is intimately related to the volatility risk premium, which in turn are determined by the model one adopts to describe the dynamics of the underlying asset returns.

In continuous-time modelling in finance, stochastic volatility (SV) models, such as the models of Hull and White (1987), Heston (1993), and many others, have attracted a great deal of attention, as they have captured successfully many of the stylized facts of financial asset returns, such as time-varying volatility, volatility clustering, and leverage effect. Among these models, the square-root SV model of Heston (1993) which belongs to the general class of affine models seems to be the most popular model. The main reason behind that is because the affine SV model of Heston (1993) provide computational tractability that leads to closed-form solutions for option pricing. Unfortunately, there is a growing literature that provides empirical evidence against affine SV models (see e.g., Chernov et al., 2003; Jones, 2003; Aït-Sahalia and Kimmel, 2007, among many others).

Recently, non-affine SV models, such as GARCH or general constant elasticity of variance (CEV)-type diffusion models, have been found to capture the dynamics of the underlying asset returns much better than the affine SV model of Heston (1993) (Aït-Sahalia and Kimmel, 2007; Jones, 2003). Particularly, the GARCH diffusion model of non-affine specification has attracted a great deal of attention in recent years in the finance literature. A number of papers have provide strong evidence for the GARCH diffusion model not only for underlying asset but also for option data (e.g., Christoffersen et al., 2010; Chourdakis and Dotsis, 2011; Wu et al., 2012; Kaeck and Alexander, 2012, 2013).

Several estimation strategies for inferring the volatility risk premium have been proposed under the affine/non-affine SV models in the literature. One estimation strategy for inferring the volatility risk premium relies on the options data (see e.g., Chernov and Ghysels, 2000; Pan, 2002; Polson and Stroud, 2003; Eraker, 2004; Cheng et al., 2008; Garcia et al., 2011; Ferriani and Pastorello, 2012). However, using options data in estimation will no doubt create an unmanageable computational burden, as it inevitably involves option valuation for SV option pricing models. Instead of dealing with options data that will complicate the estimation procedure, some authors proposed alternative approach for inferring the volatility risk premium using the CBOE Volatility Index (VIX), a volatility index for the S&P500 return (see e.g., Jone, 2003; Duan and Yeh, 2010; Chourdakis and Dotsis, 2011; Bollerslev et al., 2011). The fundamental advantage of this approach is that the value for the volatility risk premium can be inferred without directly using option prices in estimation, thus avoids costly numerical option valuations and significantly reduces the computational burden.

In this paper, we study the volatility risk premium under the non-affine GARCH diffusion model of stochastic volatility, using the Chinese Volatility Index (iVIX), a volatility index similar to the CBOE VIX, which calculated based on the Shanghai 50ETF options data. We derive the corresponding implied iVIX formula under the GARCH diffusion model, and develop an efficient importance sampling (EIS)-based joint maximum likelihood (ML) estimation method for the objective and risk-neutral parameters of the GARCH diffusion model, using joint data on the Shanghai 50ETF and iVIX index. The EIS-ML method is easy to implement and enables us to estimate the parameters of the GARCH diffusion model precisely. Since the knowledge of the estimated model parameters is not sufficient to compute the volatility risk premium, we also have to know the latent spot volatility as well. We further develop in this paper a particle filter algorithm for extracting latent volatility using joint data. Then, it allows us to infer the volatility risk premium implied by the iVIX.

To illustrate our estimation approach empirically, we apply the approach to estimate the volatility risk premium using actual data on the Shanghai 50ETF and iVIX index. We find that the volatility risk is priced by the market, and the volatility risk premium is negative, implying that investors act risk averse in the Shanghai stock market.

The rest of the paper is organized as follows. In Section 2, we propose under the objective probability measure the GARCH diffusion model, and proceed to derive the corresponding system under the risk-neutral measure and the volatility risk premium. Moreover, the corresponding implied iVIX formula under the GARCH diffusion model is derived. In Section 3, we detail the EIS-ML method and the particle filter method for the parameter and latent state variable estimation of the model. The empirical results are reported in Section 4, and we conclude in Section 5.

2. The model

We adopt the non-affine GARCH diffusion model to characterize the dynamics of the underlying asset returns, and serves the basis of volatility risk premium estimation. We describe the model under the objective probability measure in Section 2.1, and derive the corresponding system under the risk-neutral measure and volatility risk premium in Section 2.2. A derivation of the iVIX formula under the GARCH diffusion model is presented in Section 2.3.

2.1 The GARCH diffusion model

In the GARCH diffusion model, the dynamics under the objective probability measure of the underlying asset price S_t, and the associated volatility, V_t, are assumed to be given by

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_{1t} \tag{1}$$

$$dV_t = (\alpha - \beta V_t)dt + \sigma V_t[\rho dW_{1t} + \sqrt{1-\rho^2} dW_{2t}] \tag{2}$$

where μ is the mean of the underlying asset returns, α / β is the long-run mean of volatility, β is the mean reversion rate of volatility, σ is the volatility of volatility, and W_{1t} and W_{2t} are two standard Brownian motions.

2.2 Risk-neutral measure and volatility risk premium

To change the objective measure to the risk-neutral one, we need to apply the

Girsanov's theorem. Specifically, let us consider the Radon-Nikodym derivative of the objective probability measure with respect to the risk-neutral one, which is

$$\xi_t = \exp\left\{-\frac{1}{2}\int_0^t (\gamma_{1u}^2 + \gamma_{2u}^2)du - \int_0^t \gamma_{1u} dW_{1u} - \int_0^t \gamma_{2u} dW_{2u}\right\} \quad (3)$$

where $\gamma_t = (\gamma_{1t}, \gamma_{2t})'$ is the vector of the market prices of risks, return and volatility risks, respectively.

Define $dW_{1t}^* = \gamma_{1t}dt + dW_{1t}$ and $dW_{2t}^* = \gamma_{2t}dt + dW_{2t}$, such that W_{1t}^* and W_{2t}^* are two standard Brownian motions under the risk-neutral measure. Then, we can proceed to determine the risk-neutral dynamics of S_t and V_t, which can be calculated as follows:

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_{1t} = [\mu - \gamma_{1t}\sqrt{V_t}]S_t dt + \sqrt{V_t} S_t dW_{1t}^*$$

Note that the discounted asset price process is a martingale under the risk-neutral measure, we have

$$\gamma_{1t} = (\mu - r)/\sqrt{V_t}$$

where r is the risk-free interest rate. Thus,

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_{1t}^* \quad (4)$$

and

$$dV_t = [\alpha - \beta V_t - \lambda(V_t)]dt + \sigma V_t[\rho dW_{1t}^* + \sqrt{1-\rho^2}dW_{2t}^*]$$

where $\lambda(V_t) = \sigma\rho\gamma_{1t}V_t + \sigma\sqrt{1-\rho^2}\gamma_{2t}V_t$, represents the volatility risk premium. We assume that the volatility risk premium is a linear function of V_t, which is $\lambda(V_t) = \eta_0 + \eta_1 V_t$.

Then the market price of volatility risk is given by

$$\gamma_{2t} = \frac{\eta_0 + \eta_1 V_t - \sigma\rho\gamma_{1t}V_t}{\sigma\sqrt{1-\rho^2}V_t}$$

such that the risk-neutral volatility dynamics can be expressed as

$$dV_t = (\alpha^* - \beta^* V_t)dt + \sigma V_t[\rho dW_{1t}^* + \sqrt{1-\rho^2}dW_{2t}^*] \quad (5)$$

where $\alpha^* = \alpha - \eta_0$, $\beta^* = \beta + \eta_1$.

Given the parameter values under both the objective and risk-neutral measures and the latent volatility, the volatility risk premium can be obtained:

$$\lambda(V_t) = (\alpha - \alpha^*) + (\beta^* - \beta)V_t \quad (6)$$

2.3 GARCH diffusion implied iVIX formula

As the iVIX index is similar to the CBOE VIX index, according to Duan and

Yeh (2010), we have

$$\mathrm{iVIX}_t^2(\tau) = \frac{2}{\tau}\left(r\tau - E_t^Q\left[\ln\frac{S_{t+\tau}}{S_t} \right] \right) \tag{7}$$

where $\tau = 30/365$, E_t^Q is the risk-neutral conditional expectation at time t.

Eq. (4) can be used to obtain

$$E_t^Q\left[\ln\frac{S_{t+\tau}}{S_t} \right] = r\tau - \frac{1}{2}\int_t^{t+\tau} E_t^Q(V_s)ds \tag{8}$$

and from Eq. (5), for any $\beta^* \neq 0$ and $\tau > 0$, we have

$$\int_t^{t+\tau} E_t^Q(V_s)ds = \frac{\alpha^*}{\beta^*}\left(\tau - \frac{1-e^{-\beta^*\tau}}{\beta^*} \right) + \frac{1-e^{-\beta^*\tau}}{\beta^*}V_t \tag{9}$$

and for $\beta^* = 0$, we have

$$\int_t^{t+\tau} E_t^Q(V_s)ds = \frac{1}{2}\alpha^*\tau^2 + \tau V_t \tag{10}$$

Combining Eqs. (7)-(10) gives rise to a theoretical link

$$\mathrm{iVIX}_t^2(\tau) = A(\tau) + B(\tau)V_t \tag{11}$$

where

$$B(\tau) = \begin{cases} \dfrac{1-e^{-\beta^*\tau}}{\beta^*\tau}, & \beta^* \neq 0 \\ 1, & \beta^* = 0 \end{cases} \quad \text{and} \quad A(\tau) = \begin{cases} \dfrac{\alpha^*}{\beta^*}\left(1 - B(\tau)\right), & \beta^* \neq 0 \\ \dfrac{1}{2}\alpha^*\tau, & \beta^* = 0 \end{cases}$$

The above result shows that the iVIX index can be linked in closed-form to the latent volatility, V_t, for the GARCH diffusion model, thus provides a simple way to deal with the estimation challenge.

3. Estimation methodology

In this section, we focus on how to estimate the objective and risk-neutral parameters and latent state variables of the GARCH diffusion model using the Shanghai 50ETF and iVIX index. First, we describe how to estimate jointly the objective and risk-neutral parameters using the EIS-ML method. Then we illustrate how to estimate the latent state variables using the particle filter algorithm.

3.1 Joint ML estimation

Let $s_t = \ln S_t$ and $h_t = \ln V_t$. By Itô's lemma, we have

$$ds_t = (\mu - \frac{1}{2}e^{h_t})dt + e^{h_t/2}dW_{1t} \tag{12}$$

$$dh_t = (\alpha e^{-h_t} - \beta - \frac{1}{2}\sigma^2)dt + \sigma[\rho dW_{1t} + \sqrt{1-\rho^2}dW_{2t}] \tag{13}$$

In the empirical literature, the above continuous-time model must be discretized to facilitate the parameter estimation. A simple Euler discretisation leads to the following discrete-time stochastic processes

$$x_t = (\mu - \frac{1}{2}e^{h_{t-1}})\Delta + e^{h_{t-1}/2}\sqrt{\Delta}\varepsilon_t \tag{14}$$

$$h_t = h_{t-1} + (\alpha e^{-h_{t-1}} - \beta - \frac{1}{2}\sigma^2)\Delta + \sigma\sqrt{\Delta}[\rho\varepsilon_t + \sqrt{(1-\rho^2)}\eta_t] \tag{15}$$

where $x_t = s_t - s_{t-1}$ is the continuously compounded return of the underlying asset, Δ is the time interval, ε_t and η_t are independent and identically distributed (i.i.d.) standard normal random variables and uncorrelated.

In order to perform the joint estimation of objective and risk-neutral parameters, we consider the additional information provided by the iVIX index. To account for possible measurement error in the iVIX index, we assume the observed log iVIX index as follows:

$$\ln \mathrm{iVIX}_t(\tau) = \frac{1}{2}\ln[A(\tau) + B(\tau)V_t] + \delta v_t \tag{16}$$

where v_t are i.i.d. standard normal random variables and independent of ε_t and η_t, δ is the parameter corresponding to the measurement error.

It is clear that Eqs. (14)-(16) constitute a nonlinear and non-Gaussian state-space model with log volatility is the latent state variable. To estimate this model using ML method, we need to integrate out the latent state variables from the joint density of the observations and latent state variables and derive an explicit expression for the marginal likelihood of observations.

Let $\mathbf{X} = (x_1, \ldots, x_T)'$ be a vector of the observed underlying asset returns (50ETF returns), let $\mathbf{Y} = (\ln \mathrm{iVIX}_1, \ldots, \ln \mathrm{iVIX}_T)'$ be a vector of the observed log iVIX and let $\mathbf{H} = (h_1, \ldots, h_T)'$ be a vector of the latent state variables which are the log volatilities in our case. The likelihood function of the observed samples of 50ETF returns and iVIX can in principle be expressed as

$$\mathrm{L}(\mathbf{X}, \mathbf{Y}; \Theta, h_0) = \int p(\mathbf{X}, \mathbf{Y}, \mathbf{H}; \Theta, h_0) d\mathbf{H} \tag{17}$$

where $\Theta = (\mu, \alpha, \beta, \sigma, \rho, \alpha^*, \beta^*, \delta)'$, h_0 is the initial log volatility, and we will estimate along with parameter vector Θ, and $p(\mathbf{X}, \mathbf{Y}, \mathbf{H}; \Theta, h_0)$ is the joint density of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{H}$, which can be written as

$$p(\mathbf{X},\mathbf{Y},\mathbf{H};\Theta,h_0)=\prod_{t=1}^{T}p(\ln \mathrm{iVIX}_t\mid h_t,\Theta)\,p(x_t\mid h_{t-1},\Theta)\,p(h_t\mid x_t,h_{t-1},\Theta) \quad (18)$$

where $p(\ln \mathrm{iVIX}_t\mid h_t,\Theta)$ is the normal density of $\ln \mathrm{iVIX}_t$ with the conditional mean $\frac{1}{2}\ln[A(\tau)+B(\tau)V_t]$ and the conditional variance δ^2, $p(x_t\mid h_{t-1},\Theta)$ is the normal density of x_t with the conditional mean $(\mu-\frac{1}{2}e^{h_{t-1}})\Delta$ and the conditional variance $e^{h_{t-1}}\Delta$ and $p(h_t\mid x_t,h_{t-1},\Theta)$ is the normal density of h_t with the conditional mean and the conditional variance are given by

$$\mu_t=h_{t-1}+(\alpha e^{-h_{t-1}}-\beta-\frac{1}{2}\sigma^2)\Delta+\sigma\rho\frac{x_t-(\mu-\frac{1}{2}e^{h_{t-1}})\Delta}{e^{h_{t-1}/2}} \quad (19)$$

$$\sigma_t^2=\sigma^2(1-\rho^2)\Delta \quad (20)$$

Given the likelihood function L, the ML estimates of parameters of the state-space model in Eqs. (14)-(16) are then given by

$$(\hat{\Theta},\hat{h}_0)=\arg\max_{(\Theta,h_0)}\ln \mathrm{L}(\mathbf{X},\mathbf{Y};\Theta,h_0)$$

3.2 EIS to likelihood approximation

Since a typical financial time series has at least several hundreds of observations, the high-dimensional integral in the right hand of Eq. (17) rarely has analytical expression. Meanwhile, using the traditional numerical integration methods to approximate the integral is also infeasible. In order to overcome this problem, we use the Monte Carlo simulation methods.

From Eq. (18), the likelihood function in Eq. (17) can be rewritten as

$$\mathrm{L}(\mathbf{X},\mathbf{Y};\Theta,h_0)=\int\prod_{t=1}^{T}p(\ln \mathrm{iVIX}_t\mid h_t,\Theta)\,p(x_t\mid h_{t-1},\Theta)\,p(h_t\mid x_t,h_{t-1},\Theta)\,d\mathbf{H} \quad (21)$$

Let $h_t^{(s)}$ is drawn independently from the so-called natural importance sampling (NIS) density $p(h_t\mid x_t,h_{t-1}^{(s)},\Theta)$, then the corresponding NIS-Monte Carlo estimate is given by

$$\tilde{\mathrm{L}}(\mathbf{X},\mathbf{Y};\Theta,h_0)=\frac{1}{S}\sum_{s=1}^{S}\left[\prod_{t=1}^{T}p(\ln \mathrm{iVIX}_t\mid h_t^{(s)},\Theta)\,p(x_t\mid h_{t-1}^{(s)},\Theta)\right] \quad (22)$$

A primary advantage of the NIS is that it is intuitive and simple to implement. However, it turns out that the NIS estimate is highly inefficient since its sampling variance rapidly increases with the sample size T. Thus this estimate cannot be relied on practically. To overcome this drawback of the NIS, we adopt the EIS proposed by Richard and Zhang (2007). Based upon this Monte Carlo integration technique, high-dimensional integral can be evaluated quickly with high numerical accuracy.

The EIS requires an auxiliary parametric importance sampler from which samples can be obtained efficiently. Let $\{m_t(h_t \mid x_t, h_{t-1}, a_t)\}_{t=1}^{T}$ be an auxiliary sampler (i.e., EIS sampler) indexed by the auxiliary parameter $\{a_t\}_{t=1}^{T}$. The likelihood function in Eq. (21) is rewritten as

$$\mathrm{L}(\mathbf{X},\mathbf{Y};\Theta,h_0)=\int\left[\prod_{t=1}^{T}\frac{p(\ln\mathrm{iVIX}_t \mid h_t,\Theta)\,p(x_t \mid h_{t-1},\Theta)\,p(h_t \mid x_t,h_{t-1},\Theta)}{m_t(h_t \mid x_t,h_{t-1},a_t)}\times\prod_{t=1}^{T}m_t(h_t \mid x_t,h_{t-1},a_t)\right]d\mathbf{H} \tag{23}$$

The corresponding EIS-Monte Carlo estimate is then given by

$$\hat{\mathrm{L}}(\mathbf{X},\mathbf{Y};\Theta,h_0)=\frac{1}{S}\sum_{s=1}^{S}\left[\prod_{t=1}^{T}\frac{p(\ln\mathrm{iVIX}_t \mid h_t^{(s)},\Theta)\,p(x_t \mid h_{t-1}^{(s)},\Theta)\,p(h_t^{(s)} \mid x_t,h_{t-1}^{(s)},\Theta)}{m_t(h_t^{(s)} \mid x_t,h_{t-1}^{(s)},a_t)}\right] \tag{24}$$

where $h_t^{(s)}$ is drawn independently from the EIS density $m_t(h_t \mid x_t, h_{t-1}^{(s)}, a_t)$.

Following Richard and Zhang(2007), we write EIS density m_t as

$$m_t(h_t \mid x_t,h_{t-1},a_t)=\frac{k_t(h_t \mid x_t,h_{t-1},a_t)}{\chi_t(x_t,h_{t-1},a_t)} \tag{25}$$

$$\chi_t(x_t,h_{t-1},a_t)=\int k_t(h_t \mid x_t,h_{t-1},a_t)\,dh_t \tag{26}$$

where $k_t(h_t \mid x_t, h_{t-1}, a_t)$ is the density kernel. According to the state-space model in Eqs. (14)-(16), we set

$$k_t(h_t \mid x_t,h_{t-1},a_t)=p(h_t \mid x_t,h_{t-1},\Theta)\exp\{a_{1,t}h_t+a_{2,t}h_t^2\} \tag{27}$$

where $a_t=(a_{1,t},a_{2,t})$.

From Eqs. (25) and (27), we have

$$\begin{aligned}&\prod_{t=1}^{T}\frac{p(\ln\mathrm{iVIX}_t \mid h_t,\Theta)\,p(x_t \mid h_{t-1},\Theta)\,p(h_t \mid x_t,h_{t-1},\Theta)}{m_t(h_t \mid x_t,h_{t-1},a_t)}\\&=p(x_1 \mid h_0,\Theta)\,\chi_1(x_1,h_0,a_1)\times\prod_{t=1}^{T}\frac{p(\ln\mathrm{iVIX}_t \mid h_t,\Theta)\,p(x_{t+1} \mid h_t,\Theta)\,\chi_{t+1}(x_{t+1},h_t,a_{t+1})}{\exp\{a_{1,t}h_t+a_{2,t}h_t^2\}}\end{aligned} \tag{28}$$

where $p(x_{T+1} \mid h_T,\Theta)\equiv\chi_{T+1}(x_{T+1},h_T,a_{T+1})\equiv 1$. Thus, we set up the following minimization problem to minimize the EIS-Monte Carlo variance of Eq. (24):

$$(\hat{a}_t(\Theta),\hat{c}_t(\Theta)) = \arg\min_{(a_t,c_t)} \sum_{s=1}^{S} \{\ln\left[p(\ln \mathrm{iVIX}_t \mid h_t^{(s)},\Theta)\, p(x_{t+1} \mid h_t^{(s)},\Theta)\,\chi_{t+1}(x_{t+1},h_t^{(s)},\hat{a}_{t+1})\right] - c_t - a_{1,t}h_t^{(s)} + a_{2,t}(h_t^{(s)})^2\}^2 \tag{29}$$

where c_t is estimated along with the auxiliary parameter a_t.

In fact, the minimization problem described in Eq. (29) is equivalent to the following auxiliary linear regression

$$\begin{aligned}\ln p(\ln \mathrm{iVIX}_t \mid h_t^{(s)},\Theta) + \ln p(x_{t+1} \mid h_t^{(s)},\Theta) + \ln \chi_{t+1}(x_{t+1},h_t^{(s)},\hat{a}_{t+1}) \\ = c_t + a_{1,t}h_t^{(s)} + a_{2,t}(h_t^{(s)})^2 + u_t^{(s)},\ \ s = 1,\ldots,S\end{aligned} \tag{30}$$

where $u_t^{(s)}$ is the error term. Since χ_{t+1} depends on a_{t+1}, the coefficients are calculated recursively, proceeding from $t = T, T-1, ..., 1$.

In summary, it is possible to compute the likelihood function of the state-space model in Eqs. (14)-(16) for given the parameter vector (Θ, h_0), based upon the following EIS algorithm:

Step 1: Draw initial samples $\{h_1^{(s)},\ldots,h_T^{(s)}\}_{s=1}^{S}$ from the NIS sampler $\{p(h_t \mid x_t, h_{t-1},\Theta)\}_{t=1}^{T}$.

Step 2: Calculate $\hat{a}_t$ by estimating the regression model (30), working backwards from $t = T$ to $t = 1$.

Step 3: Draw new samples $\{h_1^{(s)},\ldots,h_T^{(s)}\}_{s=1}^{S}$ from the EIS sampler $\{m_t(h_t \mid x_t, h_{t-1}, a_t)\}_{t=1}^{T}$.

Step 4: Repeat Step 2 and Step 3, until a reasonable convergence of the parameters $\hat{a}_t$ is obtained.

Step 5: Calculate the likelihood approximation using

$$\hat{\mathrm{L}}(\mathbf{X},\mathbf{Y};\Theta,h_0) = \frac{1}{S}\sum_{s=1}^{S}\left[\prod_{t=1}^{T}\frac{p(\ln \mathrm{iVIX}_t \mid h_t^{(s)},\Theta)\,p(x_t \mid h_{t-1}^{(s)},\Theta)\,p(h_t^{(s)} \mid x_t, h_{t-1}^{(s)},\Theta)}{m_t(h_t^{(s)} \mid x_t, h_{t-1}^{(s)},\hat{a}_t)}\right]$$

Following Richard and Zhang (2007), a same set of Common Random Numbers (CRNs) is used to obtain the draws from the EIS sampler in order to ensure the likelihood approximation $\hat{\mathrm{L}}$ be a smooth function of the parameter vector (Θ, h_0). Typically, a reasonable convergence can be obtained after 3-5 iterations.

3.3 Particle filter

Given the parameter vector (Θ, h_0), we now illustrate how to obtain the sequence of filtered estimate of the latent log volatility h_t. One way to obtain these is by means of a particle filter (Gordon et al., 1993). For our practical filtering problem, we are interested in the filtered log volatility, $E[h_t \mid \mathcal{F}_t]$, where $\mathcal{F}_t$ denotes the information set generated by the observations $\{Z_1, \ldots, Z_t\}$, and $Z_s = (x_s, \ln \mathrm{iVIX}_s)'$ denotes the joint observations of Shanghai 50ETF return and iVIX index at time s. Suppose that $p(h_{t-1} \mid \mathcal{F}_{t-1})$ is known and we want to obtain $p(h_t \mid \mathcal{F}_t)$. First, notice that

$$p(h_t \mid \mathcal{F}_t) = \int p(h_t, h_{t-1} \mid \mathcal{F}_t) dh_{t-1} = \int \frac{p(h_t, h_{t-1} \mid \mathcal{F}_t)}{p(h_{t-1} \mid \mathcal{F}_{t-1})} dP(h_{t-1} \mid \mathcal{F}_{t-1}) \qquad (31)$$

Also, from $p(x_t, \ln \mathrm{iVIX}_t, h_t, h_{t-1} \mid \mathcal{F}_{t-1}) = p(h_t, h_{t-1} \mid \mathcal{F}_t)\, p(x_t, \ln \mathrm{iVIX}_t \mid \mathcal{F}_{t-1})$, we get

$$\begin{aligned} p(h_t, h_{t-1} \mid \mathcal{F}_t) &= \frac{p(x_t, \ln \mathrm{iVIX}_t, h_t, h_{t-1} \mid \mathcal{F}_{t-1})}{p(x_t, \ln \mathrm{iVIX}_t \mid \mathcal{F}_{t-1})} \\ &= \frac{p(x_t, \ln \mathrm{iVIX}_t \mid h_t, h_{t-1}, \Theta)\, p(h_t, h_{t-1} \mid \mathcal{F}_{t-1})}{p(x_t, \ln \mathrm{iVIX}_t \mid \mathcal{F}_{t-1})} \\ &= \frac{p(\ln \mathrm{iVIX}_t \mid h_t, \Theta)\, p(x_t \mid h_t, h_{t-1}, \Theta)\, p(h_t \mid h_{t-1}, \Theta)\, p(h_{t-1} \mid \mathcal{F}_{t-1})}{p(x_t, \ln \mathrm{iVIX}_t \mid \mathcal{F}_{t-1})} \qquad (32) \end{aligned}$$

where $p(x_t \mid h_t, h_{t-1}, \Theta)$ is the normal density of x_t with the conditional mean $(\mu - \frac{1}{2} e^{h_{t-1}})\Delta + \rho e^{h_{t-1}/2} [h_t - h_{t-1} - (\alpha e^{-h_{t-1}} - \beta - \frac{1}{2}\sigma^2)\Delta] / \sigma$ and the conditional variance $e^{h_{t-1}}(1-\rho^2)\Delta$ and $p(h_t \mid h_{t-1}, \Theta)$ is the normal density of h_t with the conditional mean $h_{t-1} + (\alpha e^{-h_{t-1}} - \beta - \frac{1}{2}\sigma^2)\Delta$ and the conditional variance $\sigma^2 \Delta$.

Plugging Eq. (32) into Eq. (31), we get

$$p(h_t \mid \mathcal{F}_t) = \int \frac{p(\ln \mathrm{iVIX}_t \mid h_t, \Theta)\, p(x_t \mid h_t, h_{t-1}, \Theta)\, p(h_t \mid h_{t-1}, \Theta)}{p(x_t, \ln \mathrm{iVIX}_t \mid \mathcal{F}_{t-1})} dP(h_{t-1} \mid \mathcal{F}_{t-1}) \qquad (33)$$

In summary, the particle filter algorithm for estimating the latent log volatility as follows:

Step 1: Given a set of random samples $\{h_{t-1}^{(1)}, \ldots, h_{t-1}^{(N)}\}$ from the probability density function $p(h_{t-1} \mid \mathcal{F}_{t-1})$.

Step 2: Draw samples $\{h_t^{(1*)}, \ldots, h_t^{(N*)}\}$ from the probability density function $p(h_t \mid h_{t-1}, \Theta)$.

Step 3: Compute the normalised weight for each sample

$$q_j = \frac{p(\ln \mathrm{iVIX}_t \mid h_t^{(j*)}, \Theta)\, p(x_t \mid h_t^{(j*)}, h_{t-1}^{(j)}, \Theta)}{\sum_{i=1}^{N} p(\ln \mathrm{iVIX}_t \mid h_t^{(i*)}, \Theta)\, p(x_t \mid h_t^{(i*)}, h_{t-1}^{(i)}, \Theta)}, \quad j = 1, \ldots, N$$

Thus define a discrete distribution over $\{h_t^{(1*)}, \ldots, h_t^{(N*)}\}$, with probability mass $\{q_1, \ldots, q_N\}$.

Step 4: Resample N times from the discrete distribution defined above to generate samples $\{h_t^{(1)}, \ldots, h_t^{(N)}\}$.

4. Empirical analysis

4.1 The data

In the empirical analysis we use daily data on the Shanghai 50ETF returns and iVIX index values from February 9, 2015 to February 5, 2016. The 50ETF returns computed are logarithmic, i.e., $x_t = \log p_t - \log p_{t-1}$, where p_t is the closing price. The sample size is 244 for both 50ETF return and iVIX index values. The joint time-series is plotted in Figure 1. The data of Shanghai 50ETF is obtained from the Wind Database of China. The data of iVIX index is obtained from the Shanghai Stock Exchange.

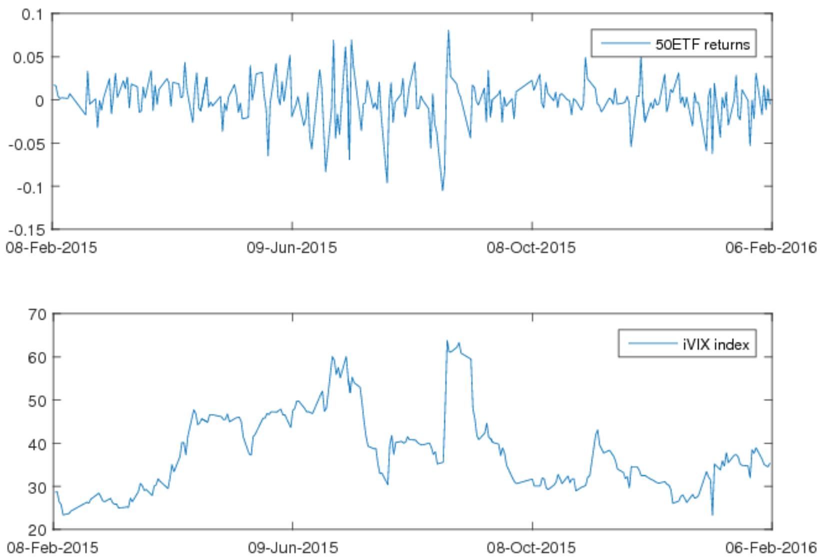

Figure 1: Joint time series of 50ETF returns and iVIX index

Summary statistics for the 50ETF returns and iVIX index values are shown in Table 1. As can be seen from the table, the 50ETF returns are skewed and leptokurtic. Jarque-Bera statistics suggests that the assumption of normality is rejected for the 50ETF return series. The summary statistics for iVIX index suggest that the 50ETF has about 37.89% annualized volatility, and the volatility ranging from 23.33% to 63.79% over the sample period. Furthermore, from Figure 1 we can observe that the 50ETF returns exhibit time-varying volatility and volatility clustering during the sample period. The results also show that the iVIX index exhibits significant characteristic of mean reversion and the behavior of highly volatile.

Table 1. Descriptive statistics of 50ETF returns and iVIX index

Data	Mean	Min	Max	Std.	Skew	Kurt	Jarque-Bera
50ETF	-0.0006	-0.1052	0.0809	0.0258	-0.6184	5.4709	77.619 (0.000)
iVIX	37.8932	23.3302	63.7886	9.0517	0.7109	3.007	20.552 (0.002)

Note: The number in parenthesis is the P-values of Jarque-Bera tests.

4.2 Estimation results

Based upon the joint data on the 50ETF returns and iVIX index values, the objective and risk-neutral parameters of the GARCH diffusion model are estimated by applying the EIS-ML method described in Section 3. Table 2 reports the estimation results.

Table 2. Estimation results

Objective parameter	μ	α	β	σ	ρ	**Log-lik**
	0.0377 (0.3672)	1.8293 (0.8456)	10.9872 (6.2433)	2.0439 (0.2253)	0.0073 (0.0655)	856.770
Risk-neutral parameter	α^*	β^*	δ			
	-0.1757 (0.0729)	0.5935 (0.2481)	0.0030 (0.0075)			

Note: The EIS-ML method is implemented by using S=32 Monte Carlo draws and 5 EIS iterations. The number in parenthesis is the asymptotic (statistical) standard error which is obtained from a numerical approximation to the Hessian. Log-lik denotes the log-likelihood value.

Our results show that the mean of the 50ETF returns is $\mu = 0.0377$. The long-run mean of the volatility is $\alpha / \beta = 0.1665$, with a fast mean-reversion speed of $\beta = 10.9872$. The estimates of the long-run mean of the volatility are higher than the unconditional sample variance of 0.1598 ($= 0.0258^2 \times 240$) (see Table 1). The estimate of the “leverage effect" parameter ρ is close to zero and not significant. The estimated objective and risk-neutral parameters (α, β) and (α^*, β^*) are quite different. It suggests that the volatility risk has mostly likely been priced by the market.

The estimate for δ, which is the parameter corresponding to the measurement error of iVIX index, closes to zero and is not significant, implying that the measurement error in iVIX can be ignored.

The estimated parameters allow us to estimate the latent volatility, V_t, via the particle filter algorithm. Figure 2 plots the estimated volatilities based upon the joint data. It can be seen from the figure that there is large fluctuations in the Shanghai stock market in 2015.

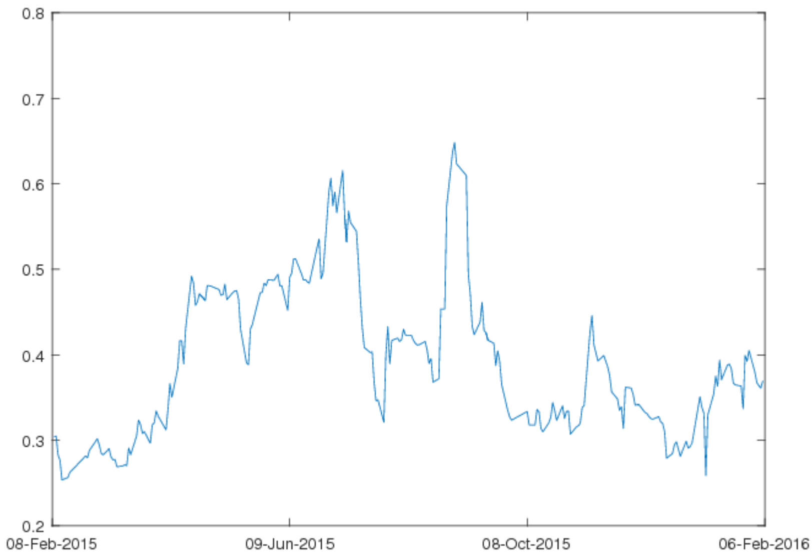

Figure 2: Estimated volatilities

As noted in Section 2.2, the estimation of the complete objective and risk-neutral parameters and spot volatility based upon the joint data allow us to compute the volatility risk premium. Therefore, we can compute the sample path for the volatility risk premium appearing in Eq. (6). The result is reported in Figure 3. It can be seen from the figure that the volatility risk is priced, and the volatility risk premium is negative during the sample period, implying that investors act risk averse in the Shanghai stock market.

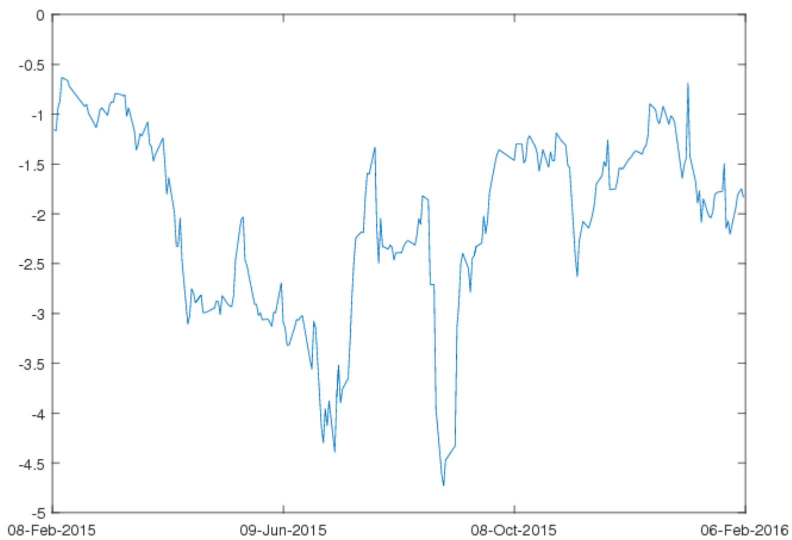

Figure 3: Estimated volatility risk premiums

5. Conclusion

In this paper, we proposed an estimation procedure for extracting the volatility risk premium in the context of the non-affine GARCH diffusion model of stochastic volatility, using joint data on the 50ETF returns and iVIX index values. The proposed estimation procedure is based on the ML method, where the likelihood function of the model is evaluated using the EIS technique. The approach is flexible and simple to implement. A theoretical iVIX formula in the GARCH diffusion model is derived. It is efficient enough to apply the EIS-ML method to the estimation of model (objective and risk-neutral) parameters from the joint data. To estimate the latent state variable, we developed a particle filter algorithm based upon the joint data. The empirical results show that the volatility risk is priced, and the volatility risk premium is negative, implying that investors act risk averse in the Shanghai stock market.

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China under Grant No. 71501001, the MOE (Ministry of Education in China) Project of Humanities and Social Sciences under Grant No. 14YJC790133, the China Postdoctoral Science Foundation (2015M580416), the Natural Science Foundation of Anhui Province of China under Grant No. 1408085QG139, and the Anhui Province College Excellent Young Talents Fund of China under Grant No. 2013SQRW025ZD.

REFERENCES

[1] **Aït-Sahalia, Y., Kimmel, R. (2007), *Maximum Likelihood Estimation of Stochastic Volatility Models;*** *Journal of Financial Economics* 83, 413-452;

[2] **Bollerslev, T., Gibson, M., Zhou, H. (2011), *Dynamic Estimation of Volatility Risk Premia and Investor Risk Aversion from Option-Implied and Realized Volatilities;*** *Journal of Econometrics* 160(1), 235-245;

[3] **Cheng, A., Gallant, A.R., Ji, C., Lee, B.S. (2008), *A Gaussian Approximation Scheme for Computation of Option Prices in Stochastic Volatility Models;*** *Journal of Econometrics* 146(1), 44-58;

[4] **Chernov, M., Ghysels, E. (2000), *A Study towards a Unified Approach to the Joint Estimation of Objective and Risk Neutral Measures for the Purpose of Options Valuation;*** *Journal of Financial Economics* 56(3), 407-458;

[5] **Chernov, M., Gallant, A.R., Ghysels, E., Tauchen, G. (2003), *Alternative Models for Stock Price Dynamics;*** *Journal of Econometrics* 116, 225-257;

[6] **Chourdakis, K. (2004), *Non-affine Option Pricing;*** *Journal of Derivatives* 11(3), 10-25;

[7] **Chourdakis, K., Dotsis, G. (2011), *Maximum Likelihood Estimation of Non-Affine Volatility Processes;*** *Journal of Empirical Finance* 18(3), 533-545;

[8] **Christoffersen, P., Jacobs, K., Mimouni, K. (2010), *Volatility Dynamics for the S&P 500: Evidence from Realized Volatility, Daily Returns, and Option Prices;*** *Review of Financial Studies* 23(8), 3141-3189;

[9] **Drimus, G.G. (2012), *Options on Realized Variance by Transform Methods: A Non-Affine Stochastic Volatility Model;*** *Quantitative Finance* 12(11), 1679-1694;

[10] **Duan, J.C., Yeh, C.Y. (2010), *Jump and Volatility Risk Premiums Implied by VIX;*** *Journal of Economic Dynamics and Control* 34, 2232-2244;

[11] **Durham, G.B. (2013), *Risk-neutral Modelling with Affine and Non-Affine Models;*** *Journal of Financial Econometrics* 11(4), 650-681;

[12] **Eraker, B. (2004),** ***Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices;*** *Journal of Finance* 59(3), 1367-1404;

[13] **Ferriani, F., Pastorello, S. (2012),** ***Estimating and Testing Non-Affine Option Pricing Models with a Large Unbalanced Panel of Options;*** *Econometrics Journal* 15, 171-203;

[14] **Garcia, R., Lewis, M.A., Pastorello, S., Renault, E. (2011),** ***Estimation of Objective and Risk-Neutral Distributions Based on Moments of Integrated Volatility;*** *Journal of Econometrics* 160(1), 22-32;

[15] **Gordon, N.J., Salmond, D.J., Smith, A.F. (1993),** ***Novel Approach to Nonlinear/Non-Gaussian Bayesian State Estimation;*** *IEE Proceedings-F* 140, 107-113;

[16] **Heston, S.L. (1993),** ***A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options;*** *Review of Financial Studies* 6(2), 327-343;

[17] **Hull, J.C., White, A.D. (1987),** ***The Pricing of Options on Asset with Stochastic Volatilities;*** *Journal of Finance* 42(2), 281-300;

[18] **Jones, C. (2003),** ***The Dynamics of Stochastic Volatility: Evidence from Underlying and Options Markets;*** *Journal of Econometrics* 116, 181-224;

[19] **Kaeck, A., Alexander, C. (2012),** ***Volatility Dynamics for the S&P 500: Further Evidence from Non-Affine, Multi-Factor Jump Diffusions;*** *European Financial Management* 36, 3110-3121;

[20] **Kaeck, A., Alexander, C. (2013),** ***Stochastic Volatility Jump-Diffusions for European Equity Index Dynamics;*** *European Financial Management* 19(3), 470-496;

[21] **Pan, J. (2002),** ***The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study;*** *Journal of Financial Economics* 63, 3-50;

[22] **Polson, N.G., Stroud, J.R. (2003),** ***Bayesian Inference for Derivative Prices;*** *Bayesian Statistics* 7, 641-650;

[23] **Richard, J.F., Zhang, W. (2007),** ***Efficient High-Dimensional Importance Sampling;*** *Journal of Econometrics* 127(2), 1385-1411;

[24] **Wu, X.Y., Ma, C.Q., Wang, S.Y. (2012),** ***Warrant Pricing under GARCH Diffusion Model;*** *Economic Modelling* 29(6), 2237-2244.