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NON-CONSTANT QUASI-HYPERBOLIC DISCOUNTING

Abstract. *This paper puts forward a non-constant quasi-hyperbolic (NQH) discount function which can control the switch point of preference reversal in a flexible way. A non-standard Hamilton-Jacobi-Bellman (HJB) equation enables us to produce time-consistent solution under stochastic non-constant quasi-hyperbolic (SNQH) discounting. The sophisticated individual, the naïve individual and the pre-committed individual are compared analytically and numerically.*

Key word*: preference reversal; time-consistent solution; sophisticated individual; naïve individual; pre-committed individual.*

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1. Introduction

Quasi-hyperbolic (**QH**) discount function, in particular *constant* QH (**CQH**[1]) discounting, has long been recognized as the corner stone of preference theory. Pan, Webb, Zank (**PWZ**) (2015) propose a QH discount function whose major advantage is capturing the switch point of preference reversal. After these endeavors, the usage of *constant* long-run discount rate ($-\ln\delta$ and ρ) to

[1] the constant quasi-hyperbolic (**CQH)** discount function includes, but not limited to, $1,\beta\delta,\beta\delta^2,\cdots$ or $1,\beta e^{-\rho},\beta e^{-2\rho},\cdots$, where $-\ln\delta$ and ρ are ***constant*** long-run discount rates. $\beta\delta^s$ and $\beta e^{-\rho s}$ are discount factors used to evaluate the payoff enjoyed at $s+1$.

aggregate time-varying preference of conflicting selves remains “intact”. The purpose of this paper is to reconcile this ***constancy*** with the fact that selves make ***irregular*** modification of their preferences in every instant.

The paper comprises three main components. First, we design a **NQH**[1] discount function. Relative to the earlier frameworks, the NQH discount function adds two new features: (i). by calibrating the non-constant long-run discount rate $r(\tau)$, the switch point between different preferences can be adjusted easily. (ii). multiple preference reversals becomes possible, as long as the assumption of non-increasing $r(\tau)$ is relaxed. In addition, by extending the Harrison-Laibson discount function (Harris and Laibson, 2013), section 2 approximates the conflicts that each self faces with the SNQH discount function.

Second, to reach the subgame perfect equilibrium in a non-cooperative sequential game, we reformulate an objective function of an SNQH discounter into a non-standard HJB. The intuition for this is that, by aggregating preferences, the SNQH discounter has no incentive to deviate from time-consistent policy. Formal difficulties have been evident since Karp (2007) pioneered the non-standard HJB method, Ekeland, Mbodji and Pirvu (2010) popularized a verification theorem, and Zou, Chen and Wedge (2014) tackled a problem under a CQH discounting. These difficulties are crystallized here since the present paper studies a sophisticated SNQH discounter who has never appeared in the literature.

Third, psychological features of different individuals (e.g. myopic and pre-committed) are explored via the deformation of the afore-mentioned HJB. In other word, our versatile discount function admits situations in which individuals can either commit or not to their initial plan over time.

[1] the **NQH** discount function can take the form of $1, \beta\theta(1), \beta\theta(2), \cdots$ where $\theta(s) = \exp\left(-\int_0^s r(\tau)\,d\tau\right)$ is the discount factor used to evaluate the payoff enjoyed at $s+1$ and $r(\tau)$ is a ***non-constant*** long-run discount rate.

2. **Preference**

This section establishes a new discount function, which integrates the NQH discount function with the split algorithm in Harrison and Laibson (2013).

2.1. Discount function

Definition 2.1 (stochastic non-constant quasi-hyperbolic discount function, SNQH**)** *The discount function with which a t-agent (decision time is t) evaluates a payoff at time s is*

$$D(t,s)=\begin{cases}\theta(s-t), & s\in[t,t+\varsigma)\\ \beta\theta(s-t), & s\in[t+\varsigma,\infty)\end{cases} \tag{2.1}$$

here (i) If enjoying her payoff at time $s\in[t,t+\varsigma)$, self t discounts the payoff with a hyperbolic discount factor $\theta(t,s)=\theta(s-t)=\exp\left(-\int_0^{s-t} r(\tau)d\tau\right)$. $\theta(s-t)$ dollar for self t is equal to one dollar for self s ($t<s<T$). $E(\varsigma)=\frac{1}{\lambda}$. (ii) The payoff enjoyed at time $s\in[t+\varsigma,\infty)$ is discounted with $\beta\theta(s-t)$. (iii) $r(\tau)$ is the long-run discount rate.□

Of particular significance is $r(\tau)$, which allows the NQH (and SNQH) discount function have non-constant discount rates at arbitrary stage/status and can be parameterized with

$$\begin{cases}\theta(s)=e^{\int_0^s -r(\tau)d\tau}\\ r(\tau)=a-\dfrac{b}{T}\tau\end{cases} \tag{2.2}$$

where T is the terminal time point, a and b are determinants of preference and exert independent, and often opposing, effects.

2.2. Preference reversal

Three examples examine how the NQH discounting, an element of the SNQH discounting, accounts for the switches between present-bias to future-bias.

2.2.1. Preference reversal

We adapt/refine Jackson and Yariv (2015) to explain the phenomenon that an impatient individual who prefers (one apple, 0 day) to (two apples, 1 day) becomes patient in selecting (two apples, 366 days) instead of (one apple, 365 days).

Definition 2.2. (**preference reversal**) *(i)* C' *and* C'' *stand for two consumption streams; (ii)* C_i *denotes activity (e.g. consumption) at time* i, $i = 0,1,2,\cdots$*; (iii)* t *represents decision time point,* s *the time point that a payoff will be enjoyed,* k *the delay (time distance) between two timed outcomes,* m *the translation of time. (iv)* V *is the discounted utility (DU) at four time points* $s = \{0, k, m, k+m\}$, $0 < k$, $0 < m$,

$$V(C') = \begin{cases} V\left(C_0', 0, \cdots\right) & if \ \ (t,s) = (0,0) \\ V\left(0, \cdots, C_m', 0, \cdots\right) & if \ \ (t,s) = (0,m) \end{cases}$$

$$V(C'') = \begin{cases} V\left(0, \cdots, C_{k+0}'', 0 \cdots\right) & if \ \ (t,s) = (0,k) \\ V\left(0, \cdots 0, C_{k+m}'', 0 \cdots\right) & if \ \ (t,s) = (0,k+m) \end{cases}$$

If, the initial preference represented by a function V

$$V\left(C_0', 0, \cdots\right) > V\left(0, \cdots, C_{k+0}'', 0 \cdots\right)$$

is opposite to the preference after a translation of m *unit time*

$$V\left(0, \cdots, C_m', 0 \cdots\right) < V\left(0, \cdots 0, C_{k+m}'', \cdots\right),$$

then, the preference is reversed. □

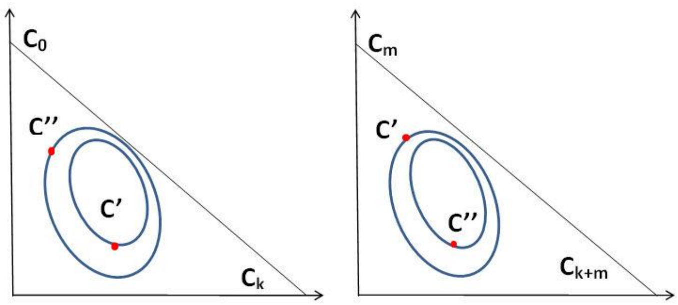

Figure 2.1 Impatient short run selves and patient long-run selves

As Fig. 2.1 illustrates, preference reversal occurs merely because the timed outcomes are translated m units into the future. At time 0, the discounter is reluctant to accept a delay of k and has a preference of the sooner reward $(C', 0)$ over the delayed reward (C'', k). Also at time 0, this discounter accepts the delay of k and prefers the delayed reward $(C'', m+k)$ over the sooner reward (C', m).

2.2.2. Immediacy bias with translation

It has been pointed out by PWZ (2015) that the split algorithm of the Harrison-Laibson discount function has restrictive requirements when expressing preference reversal. Contrary to their conclusion, we will demonstrate by three examples that the deterministic SNQH discounting (i.e. the NQH discounting) reflects nuance psychotic characteristics discounting.

Let s be the payoff-enjoying time, $\theta(s)$ be as defined as in eq.2.2, the payoff $C'=$one apple，the payoff $C''=$two apples，the translation $m=365$. In each example, the self t (t=0) is required to select payoff-enjoying time s twice. DU is computed using the formulas (cf. Laibson, 2010).

$$\begin{cases} V = u_s + \beta(\delta u_{s+1} + \delta^2 u_{s+2} + \cdots) \\ V = u_s + \beta(e^{-\rho} u_{s+1} + \beta e^{-2\rho} u_{s+2} + \cdots) \\ V = u_s + \beta(\theta(1) u_{s+1} + \theta(2) u_{s+2} + \cdots) \end{cases}$$

We can infer from the examples (see appendix A) that the NQH discounting has the following features:

(i) the preference reversal

$$(C',0) \succ_0 (C'', k) \text{ and } (C', m) \prec_0 (C'', m+k)$$

has been observed;

(ii) the demarcation point that separates periods before and after a change in preference can stay at arbitrary positions, e.g. $s=\{26,17,162\}$.

(iii) the switch point can be adjusted via a and b.

Thus the stochastic NQH discounting (i.e. the SNQH discounting) captures subtle heterogeneity even among present/future selves and allows all present/future selves be ordered in terms of their discount factors.

3. HJBs for time-consistent/time-inconsistent individuals

This section derives behavioral equations for different individuals.

3.1. Non-standard programming

Let x (resp. u) be a state trajectory (resp. a feasible control). For $R_+ = [0,\infty)$, $(x,u,s) \in R_+ \times R^n$, x and u are admissible, predictable and satisfy usual conditions. Let L and F be utility functions (for more details, see Karp, 2007; Marín-Solano and Navas, 2010; Zou, Chen and Wedge, 2014). The aim of a representative individual is to select x and u to maximum her discounted utility, namely

$$J(x,u,t) = \int_t^T D(t,s)L\big(x(s),u(s),s\big)ds + D(t,T)F(x(T),T)$$

$$d\dot{x}(s) = f\big(x(s),u(s),s\big)$$

V is a value function if

$$V(x,t) = J(x,u,t) \tag{3.1}$$

Theorem 3.1. *Let* $e^{-\int_0^{k\varepsilon} r(\tau)d\tau} = \theta(k)$. *The vale function* V *solves the following equation*

$$\begin{cases} r(T-t)V(x,t) + K(x,t) - V_t(x,t) = \max\limits_u \left[L(x,u,t) + V_x(x,t)\cdot f\big(x,u,t\big)\right] \\ K = \int_t^T \theta(s-t)\begin{bmatrix}\lambda(1-\beta)e^{-\lambda(s-t)} + \\ r(s-t) - r(T-t)\end{bmatrix} L(u^*(s))ds \end{cases} \tag{3.2}$$

where K *is determined by the marginal utility of future selves and accounts for the incentive to deviate the time-consistent policies*. □

3.2. Financial market

We consider an arbitrage-free market in a complete probability space $(\Omega, \mathcal{F}, P)$ with terminal time T, $T > 0$. The filtration $\{\mathcal{F}_t\}_{t\ge 0}$ is generated by two m -dimensional Brownian motions Z_1 and Z_2 , satisfying the usual condition of right-continuity and augmentation by P -null sets. A representative individual earns income from the low-risk asset Q_1, the high-risk asset Q_2 and the riskless asset Q_3 ,

$$dQ_1(t) = Q_1(t)\left(\upsilon dt + \sigma dZ_1\right),\ \ dQ_2(t) = Q_2(t)\left(\mu dt + \kappa dZ_2\right),\ \ dQ_3(t) = Q_3(t)\mu_0 dt$$

where μ_0 is a $n \times m$ matrix-valued interest rate process. υ $(\text{resp.}\ \mu)$ denotes a n -dimensional mean rate of return on quantitative investment fund (resp. stock). σ $(\text{resp.}\ \kappa)$ denotes a $n \times m$ matrix-valued volatility of the quantitative investment fund (resp. stock). ι is the covariance between Q_1 and Q_2 . p $(\text{resp.}\ \pi)$ is the fraction of total wealth invested in quantitative investment fund (resp. stock). Denoted by c consumption. WLOG, assuming $n = m = 1$, then the profit generating process is

$$\begin{aligned} dW(t) = &\left\{\left[\left(\upsilon - \mu_0\right) p(t) + \left(\mu - \mu_0\right)\pi(t) + \mu_0\right] W(t) - c(t)\right\} dt \\ &+ W(t)\left[\sigma p(t) dZ_1(t) + \kappa \pi(t) dZ_2(t)\right] \end{aligned} \tag{3.3}$$

Let $V \equiv V_W^{c,\pi,p}$ be a value function, U a utility function, c consumption, W wealth. An individual maximizes her utilities from life-cycle behavior c and final wealth $F(W(T),T)$,

$$V(W,t) = E\left[\int_t^T D(t,s)U(c(s),s)ds + D(t,T)F(W(T),T)\right] \tag{3.4}$$

While Yuan (2009) uses similar asset structure under constant exponential discounting preference, this paper is concerned with an intrapersonal game among an infinite number of selves.

3.3. **HJB for different individuals**

Denoted by superscripts N, P, S the naïve individual, the pre-committed individual and the sophisticated individual. From theorem 3.1, one has.

Proposition 3.2. *the sophisticated SNQH discounter obtains her time-consistent policy by solving the following HJB*

$$\begin{cases} r(T-t)V^S(W,t)+K(W,t)-\dfrac{\partial V^S(W,t)}{\partial t}= \\ \max\limits_{c,\pi,p}\left\{ \begin{array}{l} U(c(t))+\begin{bmatrix} (\mu-\mu_0)\pi(t)W(t)+ \\ (\upsilon-\mu_0)\,p(t)W(t)+ \\ \mu_0 W(t)-c(t) \end{bmatrix}\dfrac{\partial V^S(W,t)}{\partial W} \\ +\dfrac{1}{2}\begin{bmatrix} \sigma^2 p^2(t)+\kappa^2\pi^2(t) \\ +2p(t)\pi(t)\iota\sigma\kappa \end{bmatrix}W^2(t)\dfrac{\partial^2 V^S(W,t)}{\partial W^2} \end{array} \right\} \\ V^S(W,T)=F(W(T)) \end{cases} \tag{3.5}$$

where

$$K(W,t)=\int_t^T \left\{\theta(s-t)\left[\lambda(1-\beta)e^{-\lambda(s-t)}+r(s-t)-r(T-t)\right]U(c^*(s))\right\}ds \quad \square$$

Since selves of the *pre-committed* individual commit to self-t's plan (e.g. $t=0$), and selves of the *naïve* individual revise their policies at all times, one has the following proposition,

Proposition 3.3. *The policies of the naïve individual and the pre-committed individual are determined by the following HJB*

$$
\begin{cases}
r(\tau-t)V^{P,N}(W,\tau)-\dfrac{\partial V^{P,N}(W,\tau)}{\partial \tau}= \\
\max\limits_{c,\pi,p}\left\{\begin{array}{l}
U(c(\tau))+\begin{bmatrix}(\mu-\mu_0)\pi(\tau)W(\tau)+ \\ (\upsilon-\mu_0)\,p(\tau)W(\tau) \\ +\mu_0 W(\tau)-c(\tau)\end{bmatrix}\dfrac{\partial V^{P,N}(W,\tau)}{\partial W} \\
+\dfrac{1}{2}\begin{bmatrix}\sigma^2 p^2(\tau)+2\iota\, p(\tau)\pi(\tau)\sigma\kappa \\ +\kappa^2\pi^2(\tau)\end{bmatrix}W^2(\tau)\dfrac{\partial^2 V^{P,N}(W,\tau)}{\partial W^2}
\end{array}\right\} \\
V^{P,N}\left(W,T\right)=F\left(W(T)\right)
\end{cases}
\tag{3.6}
$$

The above proposition complements several recent studies (e.g. Caliendo, 2011).

4. Optimal strategies

This section assesses optimal consumption and investment of different SNQH discounters.

4.1. Consumption

4.1.1. Logarithmic utility

The utility function, the value function and the boundary conditions are

$$U(c(t))=\ln c,\quad V(W,t)=\alpha(t)\ln W+\eta(t),\quad F(W(T))=\xi_\beta \ln W \tag{4.1}$$

where $\xi(\beta)\equiv\xi_\beta$ is a function of β, $\alpha(t)$ and $\eta(t)$ are functions to be determined. In view of eq3.5, eq3.6 and eq. 4.1, we have

Table. 4.1. consumption under logarithmic utility

individuals	consumption
i. pre-committed	$c^P(t)=\dfrac{W(t)}{\dfrac{\theta(T)}{\theta(t)}\xi_\beta+\displaystyle\int_t^T\dfrac{\theta(s)}{\theta(t)}ds}$
ii. naifs	$c^N(t)=\dfrac{W(t)}{\theta(T-t)\xi_\beta+\displaystyle\int_t^T\theta(s-t)ds}$

iii. sophisti-cates	$c^S(t)=\dfrac{W(t)}{\theta(T-t)\xi_\beta-\int_t^T \dfrac{\theta(T-t)}{\theta(T-s)}\psi^S(s)ds}$ $\psi^S(s)=\int_s^T \theta(z-s)\left[\lambda(1-\beta)e^{-\lambda(z-s)}+r(z-s)-r(T-s)\right]dz-1$

Table 4.1 reveals the consumption c is positively correlated with the wealth W. Although the decision maker is a collection of selves, the policy that the "decision self" of the sophisticated individual makes is optimal to all other selves.

4.1.2. CRRA utility

The utility function, the conjectural value function and the boundary condition are

$$U(c(t))=\frac{c^{1-\gamma}}{1-\gamma},\quad V(W,t)=\alpha(t)\frac{W^{1-\gamma}}{1-\gamma},\quad F(W(T))=\xi_\beta\frac{(W(T))^{1-\gamma}}{1-\gamma} \tag{4.2}$$

Denoted by superscript CR the CRRA utility. For notational simplicity, we let

$$\delta^{P,CR}=\delta^{N,CR}=\delta^{CR}=\mu_0(1-\gamma)+\frac{1-\gamma}{2(1-\iota^2)\gamma}\left[\left(\frac{\mu-\mu_0}{\kappa}\right)^2-\frac{2\iota(\upsilon-\mu_0)(\mu-\mu_0)}{\kappa\sigma}+\left(\frac{\upsilon-\mu_0}{\sigma}\right)^2\right] \tag{4.3}$$

and

$$
\begin{aligned}
&\dot{\alpha}^S(t) + \left[\begin{array}{l} \mu_0(1-\gamma) - r(T-t) \\ + \dfrac{1-\gamma}{(1-\iota^2)2\gamma}\left(\begin{array}{l}\left(\dfrac{\mu-\mu_0}{\kappa}\right)^2 - \\ \dfrac{2\iota(\upsilon-\mu_0)(\mu-\mu_0)}{\kappa\sigma} + \left(\dfrac{\upsilon-\mu_0}{\sigma}\right)^2\end{array}\right)\end{array}\right]\alpha^S(t) \\
&= -\gamma\left(\alpha^S(t)\right)^{\frac{\gamma-1}{\gamma}} + \\
&\int_t^T \left\{\begin{array}{l} \theta(s-t)\left[\lambda(1-\beta)e^{-\lambda(s-t)} + r(s-t) - r(T-t)\right]\left(\alpha^S(s)\right)^{\frac{\gamma-1}{\gamma}}) \times \\
\exp\left(\begin{array}{l} \dfrac{(2\gamma-1)(1-\gamma)(s-t)}{2(1-\iota^2)\gamma^2} \times \\
\left(\left(\dfrac{\mu-\mu_0}{\kappa}\right)^2 - \dfrac{2\iota(\upsilon-\mu_0)(\mu-\mu_0)}{\kappa\sigma} + \left(\dfrac{\upsilon-\mu_0}{\sigma}\right)^2\right) \\
+\mu_0(s-t)(1-\gamma) - \displaystyle\int_t^s (1-\gamma)\left(\alpha^S(x)\right)^{\frac{-1}{\gamma}} dx \\
+(1-\gamma)\dfrac{(\upsilon-\mu_0)\kappa - (\mu-\mu_0)\iota\sigma}{(1-\iota^2)\gamma\sigma\kappa}\left(Z_1(s) - Z_1(t)\right) \\
+(1-\gamma)\dfrac{(\mu-\mu_0)\sigma - (\upsilon-\mu_0)\iota\kappa}{(1-\iota^2)\gamma\kappa\sigma}\left(Z_2(s) - Z_2(t)\right)
\end{array}\right)\end{array}\right\} ds
\end{aligned}
\tag{4.4}
$$

Let δ^{CR} be given as in eq.4.3. From eq3.5, eq3.6 and eq.4.2, one has

Table 4.2 consumption under isoelastic utility

individuals	consumption $c^*(t) = \alpha(t)^{\frac{-1}{\gamma}} W(t)$
i. pre-committed	$\alpha^P(t) = \dfrac{\theta(T)}{\theta(t)} e^{\delta^{CR}(T-t)}\left[\xi_\beta^{\frac{1}{\gamma}} + \displaystyle\int_t^T \left(\dfrac{\theta(s)}{\theta(T)} e^{-\delta^{CR}(T-s)}\right)^{\frac{1}{\gamma}} ds\right]^\gamma$
ii. naifs	$\alpha^N(t) = \theta(T-t) e^{\delta^{CR}(T-t)}\left[\xi_\beta^{\frac{1}{\gamma}} + \displaystyle\int_t^T \left(\dfrac{\theta(s-t)}{\theta(T-t)} e^{-\delta^{CR}(T-s)}\right)^{\frac{1}{\gamma}} ds\right]^\gamma$
iii. sophisticates	$\alpha^S(t)$ satisfies eq.4.4

Table 4.2 disentangles the relative risk aversion and analyzes the effects of key structural parameters on equilibrium consumption.

4.1.3. CARA utility

The utility function, the conjectural value function and the boundary conditions are

$$U(c(t)) = -\frac{1}{\gamma}e^{-\gamma c},\quad V(W,t) = -\xi_\beta e^{-\gamma(\alpha(t)+\eta(t)W)},\quad F(W,T) = -\xi_\beta e^{-\gamma W} \tag{4.5}$$

Denoted by superscript CA the CARA utility. In order to compress notation, we let

$$\begin{cases} \eta^{P,CA}(t) = \eta^{N,CA}(t) = \eta^{S,CA}(t) = \eta^{CA}(t) = \dfrac{\mu_0}{1+(\mu_0-1)e^{-\mu_0(T-t)}} \\ \delta^{P,CA}(s) = \delta^{N,CA}(s) = \delta^{CA}(s) = \eta^{CA}(s) - \eta^{CA}(s)\ln\left(a\gamma\eta^{CA}(s)\right) - \\ \qquad \dfrac{1}{2(1-\iota^2)}\left[\left(\dfrac{\mu-\mu_0}{\kappa}\right)^2 - \dfrac{2\iota(\upsilon-\mu_0)(\mu-\mu_0)}{\kappa\sigma} + \left(\dfrac{\upsilon-\mu_0}{\sigma}\right)^2\right] \end{cases} \tag{4.6}$$

and

$$\begin{aligned} \dot{\alpha}^S(t) - \eta^{CA}(t)\alpha^S(t) &= \frac{-r(T-t)+\eta^{CA}(t)-\eta^{CA}(t)\ln\left(a\gamma\eta^{CA}(t)\right)}{\gamma} \\ &- \frac{1}{(1-\iota^2)2}\left(\left(\frac{\mu-\mu_0}{\kappa}\right)^2 - \frac{2\iota(\upsilon-\mu_0)(\mu-\mu_0)}{\kappa\sigma} + \left(\frac{\upsilon-\mu_0}{\sigma}\right)^2\right) \\ &- \frac{1}{\gamma}\int_t^T \left[\begin{array}{l} \theta(s-t)\left[\lambda(1-\beta)e^{-\lambda(s-t)} + r(s-t) - r(T-t)\right]\times \\ \exp\left(-\gamma\left(\begin{array}{l} \alpha(s)-\alpha(t)+\displaystyle\int_t^s \eta^{CA}(\tau)B(\tau)d\tau \\ +\displaystyle\int_t^s \eta^{CA}(\tau)C(\tau)d\bar{Z}_1 + \int_t^s \eta^{CA}(\tau)F(\tau)d\bar{Z}_2 \end{array}\right)\right) \end{array}\right] ds \end{aligned} \tag{4.7}$$

where

$$\begin{cases} B(\tau)=\dfrac{1}{\left(1-\iota^2\right)\gamma\eta^{CA}(\tau)}\left(\left(\dfrac{\mu-\mu_0}{\kappa}\right)^2+\left(\dfrac{\upsilon-\mu_0}{\sigma}\right)^2-\dfrac{2\iota\left(\mu-\mu_0\right)\left(\upsilon-\mu_0\right)}{\kappa\sigma}\right) \\ \quad +\dfrac{1}{\gamma}\ln\left(a\gamma\eta^{CA}(\tau)\right)-\alpha^S(\tau) \\ C(\tau)=\dfrac{1}{\left(1-\iota^2\right)\gamma\eta^{CA}(\tau)}\left(\dfrac{\upsilon-\mu_0}{\sigma}-\iota\dfrac{\mu-\mu_0}{\kappa}\right) \\ F(\tau)=\dfrac{1}{\left(1-\iota^2\right)\gamma\eta^{CA}(\tau)}\left(\dfrac{\mu-\mu_0}{\kappa}-\iota\dfrac{\upsilon-\mu_0}{\sigma}\right) \end{cases}$$

Let δ^{CA} and η^{CA} be given as in eq.4.6. From eq3.5, eq3.6 and eq. 4.5, one has,

Table 4.3 consumption under CARA utility

individuals	consumption
	$c^*(t)=\dfrac{1}{-\gamma}\ln\left(\xi_\beta\gamma\eta^{CA}(t)\right)+\alpha(t)+\eta^{CA}(t)W(t)$
i. pre-committed	$\alpha^P(t)=-\int_t^T\dfrac{1}{\gamma}\left[\delta^{CA}(s)-r(s)\right]e^{-\int_t^s\eta^{CA}(m)dm}ds$
ii. naifs	$\alpha^N(t)=-\int_t^T\dfrac{1}{\gamma}\left[\delta^{CA}(s)-r(s-t)\right]e^{-\int_t^s\eta^{CA}(m)dm}ds$
iii. sophisti-cates	α^S satisfies eq.4.7

Table 4.3 shows changes in the individual's consumption operate through three channels: the discount function, the absolute risk aversion and the wealth.

4.2. Investment

By same token as section 4.1, one can derive the optimal fractions of the investment in the wealth.

Table 4.4. Investment Ratio

utility	optimal stock π and fund p
log	$\pi^*=\dfrac{1}{\left(1-\iota^2\right)\kappa}\left(\dfrac{\mu-\mu_0}{\kappa}-\iota\dfrac{\upsilon-\mu_0}{\sigma}\right),\ p^*=\dfrac{1}{\left(1-\iota^2\right)\sigma}\left(\dfrac{\upsilon-\mu_0}{\sigma}-\iota\dfrac{\mu-\mu_0}{\kappa}\right)$

CRRA	$\pi^* = \frac{1}{(1-\iota^2)\gamma\kappa}\left(\frac{\mu-\mu_0}{\kappa} - \iota\frac{\upsilon-\mu_0}{\sigma}\right),\ p^* = \frac{1}{(1-\iota^2)\gamma\sigma}\left(\frac{\upsilon-\mu_0}{\sigma} - \iota\frac{\mu-\mu_0}{\kappa}\right)$
CARA	$\pi^* = \frac{\frac{\mu-\mu_0}{\kappa} - \iota\frac{\upsilon-\mu_0}{\sigma}}{(1-\iota^2)W(t)\gamma\eta^{CA}(t)\kappa},\ p^* = \frac{\frac{\upsilon-\mu_0}{\sigma} - \iota\frac{\mu-\mu_0}{\kappa}}{(1-\iota^2)W(t)\gamma\eta^{CA}(t)\sigma}$

Table 4.4 reveals that (i) the fraction of investment in wealth π (resp. p) is an decreasing functions of its volatility κ (resp. σ), and an increasing functions of its asset return μ (resp. υ); (ii) π / p is negatively related to risk aversion coefficient γ; (iii) the optimal investment π / p is 0 when variance between two risky assets ι is -1 or 1. (iv) under CARA utility, the fraction of investment is contingent on wealth.

4.3. Simulation

In conducting the numerical analysis of the model, we use $a = 0.09$, $b = 0.006$, $\xi_\beta = 1.006$, $\beta = 0.2$ and $\lambda = 0.3$ unless denoted by figures.

4.3.1 Sophisticated individual

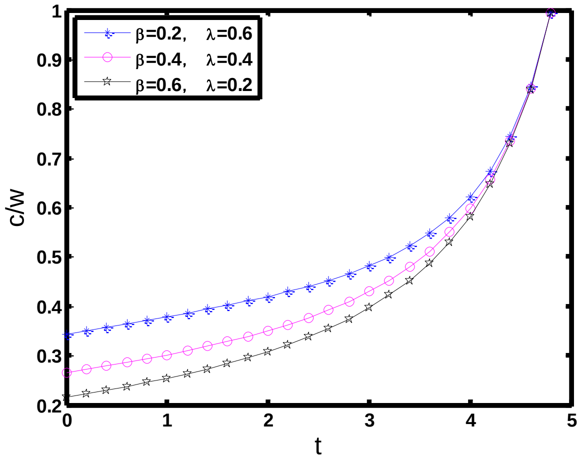

Figure 4.1. c/w ratio of the sophisticated individual by β

Figure 4.1 describes the responses of consumption curve to the quasi-hyperbolic parameters. The consumption-wealth (c/w) ratio is negatively related with the arrival rate of future λ, and positively related with immediacy-bias parameter β.

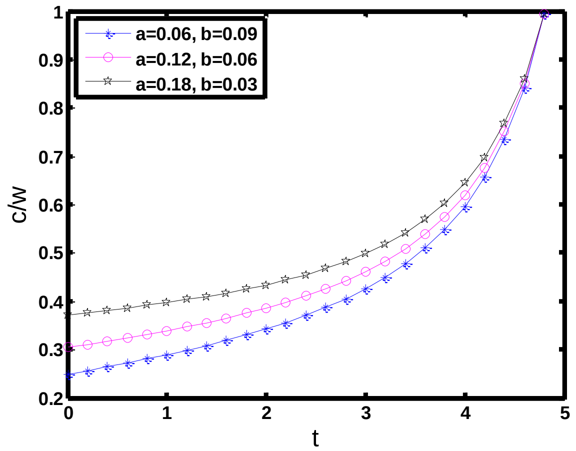

Figure 4.2. c/w ratio of the sophisticated individual by a and b

Figure 4.2 shows that the c/w ratio increases/decreases by a/b. Therefore our framework can categorize disturbances according to whether they were short-run or long-run, anticipated or unanticipated.

4.3.2 Behavioral analysis

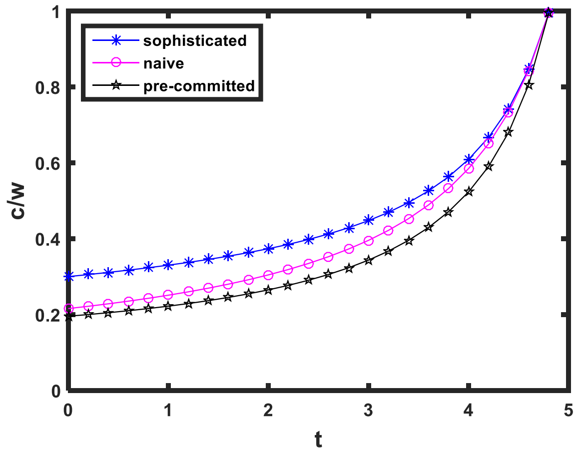

Figure 4.3. c/w ratios of the sophisticated individual, the naïve individual and the pre-committed individuals

Fig. 4.3 describes the idiosyncratic reactions of individuals. The pre-committed individual can resist temptation of overconsumption, while the sophisticated individual is affected by her preference parameters (e.g. λ, β, a, b).

The marginal propensity to consume warrants closer scrutiny.

Table 4.5 comparison of numerical simulation and analytic analysis

individuals	c/w ratio	source
i. pre-committed VS naïve	$c/W\big\|^{P} \le c/W\big\|^{N}$	fig.4.3
	$c/W\big\|^{P} \le c/W^{N}$	table. 4.1
ii. sophisticated VS naïve	$c/W\big\|^{S}_{\beta=0.2,\lambda=0.3} \ge c/W\big\|^{N}$	fig.4.3
	$c/W^{S}_{\lambda=0,\text{ or }\beta=1} \le c/W\big\|^{N}$	table. 4.1
iii. sophisticated VS pre-committed	$c/W\big\|^{S}_{\beta=0.2,\lambda=0.3} \ge c/W\big\|^{P}$	fig.4.3
	$c/W\big\|^{S}_{\lambda=0,\text{ or }\beta=1} \ge c/W\big\|^{P}$, for $t \ge T-s$ $c/W\big\|^{S}_{\lambda=0,\text{ or }\beta=1} \le c/W\big\|^{P}$, for $t \le T-s$	table. 4.1

Table 4.5 summarizes the equilibrium of various individuals. (i). To the extent that all selves discipline themselves by following self t's (t=0) plan, the pre-commitment mechanisms stabilizes the individual's rate of consumption. (ii) the consumption ratio of the sophisticated individual is larger than that of the naive individual. (iii) a decrease in bias parameter β has a strong positive effect on the consumption of the sophisticated individual.

5. Conclusion

Our paper designs the stochastic NQH discount function and finds that: (i) a decrease/increase in a/b will bring about earlier arrival of preference reversal; the stochastic NQH can have more than one instance of preference changes, if the assumption of non-increasing function in $r(\tau)$ is relaxed. (ii) consumption increases/decrease by λ / β. (iii) the sophisticated individual acts optimally and achieves a high and unified consumption trajectory.

REFERENCES

[1] **Caliendo, F. N. (2011),** ***Time-inconsistent Preferences and Social Security: Revisited in Continuous Time*****.** *Journal of Economic Dynamics & Control***, 35:668–675;**

[2] **Ekeland, I., O. Mbodji and T.A. Pirvu (2010),** ***Time Consistent Portfolio Managemen*****t.** *SIAM Journal on Financial Mathematics*, 3:1–32;

[3] **Harris, C. and D. Laibson (2013),** ***Instantaneous Gratification***. *Quarterly Journal of Economics*, 128 (1): 205–248;

[4] **Jackson M. O. and L. Yariv** ***(2015), Collective Dynamic Choice: The Necessity of Time Inconsistency***. *Microeconomic*s, 7(4): 150–178;

[5] **Karp, L. (2007),** ***Non-Constant Discounting in Continuous Time***. *Journal of Economic Theory*, 132 (1):557–568;

[6] **Laibson, D. (2010), Instant Gratification, Multiple Selves, & Self-Control,** note;

[7**] Marín-Solano, J. and J. Navas (2010),** ***Consumption and Portfolio Rules for Time-inconsistent Investors*****.** *European Journal of Operational Research*, 201: 860-872;

[8] **Pan, J., C. S. Webb and H. Zank. (2015),** ***An Extension of QH Discounting to Continuous Time***. *Games and Economic Behavior*, 89: 43–55;

[9] **Yuan, N. (2008),** ***Liquidity and Asset Pricing***. PhD thesis in Beijing University;

[10] **Zou, Z., S. Chen and L. Wedge (2014),** ***Finite Horizon Consumption and Portfolio Decisions with Stochastic Hyperbolic Discounting***. *Journal of Mathematical Economics*, 52: 70–80.

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Appendix A. Three examples of preference reversal

In this appendix we let "DU" stand for discounted utility, "prefer. " represent preference. We will prove that the NQH discounting sustains frequent and abrupt preference reversals.

Table A1. CQH VS NQH (delayed rewards: 1 day, i.e. $k=1$)

discount function			$1,\beta\delta,\beta\delta^2,\cdots$	$1,\beta\theta(1),\beta\theta(2),\cdots$
source			canonical	this paper
para-meters	mutual		$u=10,\beta=0.5,\theta(1)=\delta^1\approx 0.5$	
	special			$a=0.695,b=\frac{0.03}{400}$
short-run	DU	1 $u,(t,s)=(0,0)$	10	10
		2 $u,(t,s)=(0,1)$	4.99	4.99
	preference		present	present
long-run	DU	1 $u,(t,s)=(0,365)$	2.5×0.50^{364}	5×10^{-108}
		2 $u,(t,s)=(0,366)$	2.49×0.50^{364}	5.1×10^{-108}
	preference		present	future
prefer. reversal	position of a threshold		no	$s=26$
	threshold occur earlier		no	$a\downarrow$ *or* $b\uparrow$

Note: short-run (resp. long-run) preference is tested based on selection from $s=\{0,1\}$ (resp. $s=\{365,366\}$)

Table A2 CQH VS NQH (delayed rewards: 1 week, i.e. $k=7$)

discount function			$1,\beta\delta,\beta\delta^2,\cdots$	$1,\beta\theta(1),\beta\theta(2),\cdots$
source			canonical	this paper
para-meters	mutual		$u=10,\beta=0.5,\delta^1\equiv\theta(1)\approx 0.90$	
	special			$a=0.1,b=\frac{0.03}{400}$
short-run	DU	1 $u,(t,s)=(0,0)$	10	10
		2 $u,(t,s)=(0,7)$	4.97	4.97
	preference		present	present
long-run	DU	1 $u,(t,s)=(0,365)$	4.52×0.9^{364}	1.04×10^{-13}
		2 $u,(t,s)=(0,372)$	4.49×0.9^{364}	1.25×10^{-13}
	preference		present	future
prefer. reversal	position of a threshold		no	$s=17$
	threshold occur earlier		no	$a\downarrow$ or $b\uparrow$

Note: short-run (resp. long-run) preference is tested based on selection from $s=\{0,7\}$ (resp. $s=\{365,372\}$)

Table A3 Table 2.2 CQH VS NQH (delayed rewards: 30 days, i.e. $k=30$)

discount function		$1,\beta e^{-\rho},\beta e^{-2\rho},\cdots$	$1,\beta\theta(1),\beta\theta(2),\cdots$
source		canonical	this paper
para-meters	mutual	$u=10,\beta=0.5,e^{-\rho}=\theta(1)\approx 0.955$	
	special	$\rho=0.046$	$a=0.046,b=\frac{0.03}{400}$

short-run	D U	1 $u,(t,s)=(0,0)$	10	10
		2 $u,(t,s)=(0,30)$	2.52	2.60
	preference		present	present
long-run	D U	1 $u,(t,s)=(0,365)$	2.55×10^{-7}	3.77×10^{-5}
		2 $u,(t,s)=(0,385)$	2.04×10^{-7}	5.28×10^{-5}
	preference		present	future
prefer. reversal	position of a threshold		no	$s=162$
	threshold occur earlier		no	$a\downarrow$ or $b\uparrow$

Note: short-run (resp. long-run) preference is tested based on selection from $s=\{0,30\}$ (resp. $s=\{365,385\}$)

Appendix B. Proof of theorem 3.1

Proof. the time period $[0,T]$ is divided into N partitions such that $N=\frac{T}{\varepsilon}$ and $dt=\varepsilon$. Each period lasts for ε units of time. The time is denoted as $t=j\varepsilon$, $s=i\varepsilon$, $\theta_i=\theta(i\varepsilon)$. In addition, $V_j=V(W_j,j\varepsilon)$, $V_N=V(N\varepsilon)=F(W(T),T)$.

Discretizing eq.3.1 gives rise to

$$V_j=\sum_{i=0}^{N-j-1}D(0,i\varepsilon)L(x_{i+j},u_{i+j},(i+j)\varepsilon)\varepsilon+\beta\theta_{N-j}V_N\varepsilon \tag{B1}$$

Discretizing eq.3.1 and pushing it forward leads to

$$V_{j+1}=\sum_{i=1}^{N-j-1}D(0,(i-1)\varepsilon)L(x_{i+j},u_{i+j},(i+j)\varepsilon)\varepsilon+\beta\theta_{N-j-1}V_N\varepsilon \tag{B2}$$

A simple induction based on eq.B1 and eq.B2 yields eq.3.2.□