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A NEW METHOD FOR FINANCIAL DECISION MAKING UNDER INTUITIONISTIC LINGUISTIC ENVIRONMENT

***Abstract**. Intuitionistic linguistic set is an effective tool to deal with uncertainty. In this paper we develop a new method for financial decision making problems under intuitionistic linguistic environment and introduce the induced intuitionistic linguistic ordered weighted averaging distance (IILOWAD) operator. The main advantage of this operator is that it is able to consider complex attitudinal characters of the decision-maker by using order-inducing variables in the aggregation of the Hamming distance. Moreover, it is able to deal with uncertain environments where the information is very imprecise and can be assessed with intuitionistic linguistic numbers. We studied some of IILOWAD's main properties and different particular cases. We also develop an application of the new approach in a financial decisionmaking problem concerning the selection of civil engineering public project.*

***Keywords**:Distance measures, Induced aggregation operators, intuitionistic linguistic numbers, decision making.*

JEL Classification: D81, M12, M51

1. Introduction

Decision making is an important part of modern decision science. It has been extensively applied to various areas such as society, economics and management, military and engineering technology. In the literature, many different ways and methods are developed to solve the decision process. Among them, the most widely used one is distance measures, including the Hamming distance measure, the Euclidean distance and the Minkowski distance. The main advantage of using distance measures in decision making is that we can compare the alternatives of the problem with some ideal results (Gil-Aluja, 1999). Over the past several decades, a variety of extensions of the above distance measures have been developed. One of interesting extension is the one that uses aggregation operators in the distance measures. For example, Merigó and Gil-Lafuente (2010) have suggested the use of the ordered weighted averaging (OWA) operator (Yager, 1988) in the Hamming distance obtaining the ordered weighted averaging distance

(OWAD) operator. Merigó and Casanovas (2010) developed a linguistic ordered weighted averaging distance (LOWAD) operator by using OWA operator in the linguistic Hamming distance. Xu and Chen (2008) defined the ordered weighted distance (OWD) measure, which includes a variety of well-known distance measures and aggregation operators. Zeng and Su (2011) introduced the intuitionistic fuzzy ordered weighted distance (IFOWD) operator and studied its application in group decision making. On the basis of the idea of the induced OWA (IOWA) operator (Yager and Filev, 1999), Merigó and Casanovas (2011a) presented an induced ordered weighted averaging distance (IOWAD) operator that extends the Hamming distance measure and a reordering of arguments that depends on order-inducing variables. The IOWAD generalizes the OWAD operator and provides a parameterized family of distance aggregation operators between the maximum and the minimum distance. The main advantage of the IOWAD operator is that it is able to deal with complex attitudinal characters (or complex degrees of orness) in the decision process by using order-inducing variables. Furthermore, they also extended this approach by using the Euclidean distance (Merigó and Casanovas, 2011b) and the Minkowski distance (Merigó and Casanovas, 2011c). For further research on the use of the OWA and IOWA in kinds of distance measures, see for example, Merigó and Gil-Lafuente (2011a, 2011b, 2012), Xu and Xia (2011), Yager (2010) and Zeng and Su (2012).

Because the objects are fuzzy and uncertain, the available information involved in decision problems are not always expressed as real numbers, and sometimes it is better suited to use another approach to deal with this information such as interval numbers (Moore, 1996), fuzzy set (Zadeh, 1965; Kaufmann, 1975), intuitionistic fuzzy set (IFS) (Atanassov, 1986) and linguistic information (Herrera and Herrera-Viedma, 2000). Among all the tools, the intuitionistic fuzzy set (IFS) proposed by Atanassov (1986), considers not only a membership degree but also a non-membership degree, which is more appropriate to deal with the uncertainty and vagueness. The IFS has received more and more attention since its appearance (Boran et al., 2009; Li, 2008; Szmidt and Kacprzyk, 2003; Tan and Chen, 2010; Wei, 2010a, 2010b; Xu et al., 2010; Xu and Wang, 2012; Xu, 2007, 2011; Ye, 2010; Zeng and Su, 2011).

However, in real decision making, it is difficult for decision makers to provide exact numbers for the membership and non-membership degrees of an intuitionistic fuzzy set while it is easy to provide linguistic assessment values. On the basis of the intuitionistic fuzzy set and the linguistic assessment set, Wang and Li (2010) introduced the concept of intuitionistic linguistic set (LNS), whose basic elements are intuitionistic linguistic numbers (ILNs). The LNS can overcome the defects for intuitionistic fuzzy set which can only roughly represent criteria's membership and non-membership to a particular concept, such as ''good'' and ''bad'', etc., and for linguistic variables which usually implies that membership degree is 1, and the non-membership degree and hesitation degree of decision makers can not be expressed. In addition, Liu (2013b) further analyzed the advantages of the intuitionistic linguistic set, and developed some intuitionistic linguistic aggregation operators including the intuitionistic linguistic generalized dependent ordered weighted average (ILGDOWA) operator and the intuitionistic linguistic generalized dependent hybrid weighted aggregation (ILGDHWA) operator. Considering the interval situations, some authors proposed the interval-valued intuitionistic uncertain linguistic set (Liu, 2013a) and intuitionistic uncertain linguistic set (Liu and Fang, 2012). Su et al. (2014) introduced the intuitionistic linguistic ordered weighted averaging distance (ILOWAD)operator and studied its application in multi-person decision making problem.

From above analysis, we can see that intuitionistic linguistic set is a very useful tool to deal with uncertainty, and the study on the decision making problems with intuitionistic linguistic information has just started. Thus it is very necessary to develop some new methods to deal with the intuitionistic linguistic information. So, based on the IOWAD operator proposed by Merigó and Casanovas (2011a), in this paper, we present a new intuitionistic linguistic aggregation operator, called the induced intuitionistic linguistic OWA distance (IILOWAD) operator. It is an aggregation operator that uses the IOWA operator, distance measures and uncertain information represented in the form of intuitionistic linguistic numbers in the same formulation. With this generalization, we obtain a wide range of intuitionistic linguistic aggregation distance operators such as the max intuitionistic linguistic distance, the min intuitionistic linguistic distance, the intuitionistic linguistic normalized Hamming distance (ILNHD), the intuitionistic linguistic weighted Hamming distance (ILWHD) and the intuitionistic linguistic OWA distance (ILOWAD) operator. We also develop an application of the new approach in a financial decision making problem regarding selection of investments. The main advantage of this model in selection of investments is that it can assess uncertain situations with intuitionistic linguistic information and it gives a more complete view of the problem to the decision maker because it considers a wide range of intuitionistic linguistic aggregation operators. Therefore, the decision maker will use the particular cases that are in accordance with its interests.

This paper is organized as follows. In Section 2, we briefly review some basic concepts about intuitionistic linguistic set, the IOWA operator and the IOWAD operator. Section 3 presents the IILOWAD operator and Section 4 analyzes a wide range of particular cases. Section 5 presents an illustrative example and Section 6 summarizes the main conclusions found in the paper.

2. Preliminaries

This section briefly reviews the intuitionistic linguistic set, the IOWA operator and the IOWAD operator.

2.1 The linguistic set

The purpose of clustering methods is to group similar elements together. The similarity is established through specific distance metrics, based on which similarity or distance matrix are computer (Aggarwal, 2013). Afterward, clustering algorithms interpret the matrix and create clusters. There are three main clustering methods categories: partitional methods, hierarchical methods and quartet methods.

2.1.1 Hierarchical clustering

The linguistic approach is an approximate technique, which represents qualitative aspects as linguistic values by means of linguistic variables. For computational convenience, let $S=\{s_\alpha | \alpha = 0,1,...,l-1\}$ be a finite and totally ordered discrete term set, where l is the odd value and s_α represents a possible value for a linguistic variable. For example, when $l=9$, a set S could be given as follows:

$S=\{s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$={extremely poor, very poor, poor, slightly poor, fair, slightly good, good, very good, extremely good}

In these cases, it is usually required that there exist the following (Herrera and Herrera-Viedma, 2000; Xu, 2005):

1) A negation operator: $Neg(s_i)=s_{l-i}$;

2) The set is ordered: $s_i \le s_j$, if and only if $i \le j$;

3) Maximum operator: $\max(s_i, s_j)=s_i$, if $i \ge j$;

4) Minimum operator: $\min(s_i, s_j)=s_i$, if $i \le j$.

In order to preserve all the given information, Xu (2005) extended the discrete term set S to a continuous term set $\bar{S}=\{s_\alpha | \alpha \in [0,l]\}$, where, if $s_\alpha \in S$, then we call s_α the original term, otherwise, we call s_α the virtual term. In general, the decision maker uses the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in the actual calculation (Xu, 2005).

Consider any two linguistic terms $s_\alpha, s_\beta \in \bar{S}$, and $\mu > 0$, the operations are defined as follows:

1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$;

2) $\mu s_\alpha = s_{\mu\alpha}$;

3) $s_\alpha / s_\beta = s_{\alpha/\beta}$

2.2 The intuitionistic linguistic set(ILS)

Definition 1. (Wang and Li, 2010). An ILS A in X is defined as

$$A=\left\{\left\langle x\left[h_{\theta(x)},\left(\mu_A(x),v_A(x)\right)\right]\right\rangle\middle|x\in X\right\} \tag{1}$$

Here $h_{\theta(x)}\in\bar{S}$, and numbers $\mu_A(x)$ and $v_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to linguistic index $h_{\theta(x)}$, $0\le\mu_A(x)+v_A(x)\le 1$, for all $x\in X$.

For each ILS A in X, if

$$\pi_A(x)=1-\mu_A(x)-v_A(x),\ \forall x\in X \tag{2}$$

then $\pi_A(x)$ is called the indeterminacy degree or hesitation degree of x to linguistic Index $h_{\theta(x)}$.

Definition 2. (Wang and Li, 2010). Let $A=\left\{\left\langle x\left[h_{\theta(x)},\left(\mu_A(x),v_A(x)\right)\right]\right\rangle\middle|x\in X\right\}$ be ILS, the ternary group $\left\langle h_{\theta(x)},\left(\mu_A(x),v_A(x)\right)\right\rangle$ is called an intuitionistic linguistic number (ILN), and A can also be viewed as a collection of the ILN. So, it can also be expressed as $A=\left\{\left\langle h_{\theta(x)},\left(\mu_A(x),v_A(x)\right)\right\rangle\middle|x\in X\right\}$. In addition, $\pi_A(x)=1-\mu_A(x)-v_A(x)$ represents the hesitancy degree, and it can also be called the intuitionistic linguistic fuzzy degree. For convenience, denote an ILN by $\tilde{a}=\left\langle s_{\theta(a)},\left(\mu(a),v(a)\right)\right\rangle$, where $\mu(a),v(a)\ge 0$, $\mu(a)+v(a)\le 1$.

Let $\tilde{a}_1=\left\langle s_{\theta(a_1)},\left(\mu(a_1),v(a_1)\right)\right\rangle$ and $\tilde{a}_2=\left\langle s_{\theta(a_2)},\left(\mu(a_2),v(a_2)\right)\right\rangle$ be two ILNs and $\lambda\ge 0$, then the operations of ILNs are defined as follows (Wang and Li, 2010; Liu, 2012b).:

1) $\tilde{a}_1+\tilde{a}_2=\left\langle s_{\theta(a_1)+\theta(a_2)},\left(1-\left(1-\mu(a_1)\right)\left(1-\mu(a_2)\right),v(a_1)v(a_2)\right)\right\rangle$;

2) $\tilde{a}_1\otimes\tilde{a}_2=\left\langle s_{\theta(a_1)\times\theta(a_2)},\left(\mu(a_1)\mu(a_2),v(a_1)+v(a_2)-v(a_1)v(a_2)\right)\right\rangle$;

3) $\lambda\tilde{a}_1=\left\langle s_{\theta(a_1)^\lambda},\left(1-\left(1-\mu(a_1)\right)^\lambda,\left(v(a_1)\right)^\lambda\right)\right\rangle$.

Definition 3. (Wang and Li, 2010; Liu, 2013b). Let $\tilde{a}_1=\left\langle s_{\theta(a_1)},\left(\mu(a_1),v(a_1)\right)\right\rangle$ and $\tilde{a}_2=\left\langle s_{\theta(a_2)},\left(\mu(a_2),v(a_2)\right)\right\rangle$ be any two ILNs, then the normalized Hamming distance between $\tilde{a}_1$ and $\tilde{a}_2$ is defined as follows:

$$d_{ILHD}(\tilde{a}_1,\tilde{a}_2)=\frac{1}{2(l-1)}\times\left(\left|\left(1+\mu(a_1)-v(a_1)\right)\theta(a_1)-\left(1+\mu(a_2)-v(a_2)\right)\theta(a_2)\right|\right) \tag{3}$$

In order to compare two intuitionistic linguistic numbers, Wang and Li (2010) defined the expected value, score function and the accuracy function of ILN as following:

Definition 4. Let $\tilde{a}_1 = \left\langle s_{\theta(a_1)}, \left(\mu(a_1), v(a_1)\right)\right\rangle$ be an ILN, the expected value $E\left(\tilde{a}_1\right)$ and score function $S\left(\tilde{a}_1\right)$ of an ILN $\tilde{a}_1$ can be represented as follows:

$$E\left(\tilde{a}_1\right) = s_{\theta(a_1)\times[\mu(a_1)+\frac{1}{2}(1-\mu(a_1)-v(a_1))]} \tag{4}$$

$$S\left(\tilde{a}_1\right) = \frac{\theta(a_1)}{l-1} \times \left[\mu(a_1) + \frac{1}{2}\left(1-\mu(a_1)-v(a_1)\right)\right] \tag{5}$$

Definition 5. Let $\tilde{a}_1 = \left\langle s_{\theta(a_1)}, \left(\mu(a_1), v(a_1)\right)\right\rangle$ be an ILN, an accuracy function $H\left(\tilde{a}_1\right)$ of an ILN $\tilde{a}_1$ can be represented as follows:

$$H\left(\tilde{a}_1\right) = \frac{\theta(a_1)}{l-1} \times \left(\mu(a_1) + v(a_1)\right) \tag{6}$$

Definition 6. If $\tilde{a}_1 = \left\langle s_{\theta(a_1)}, \left(\mu(a_1), v(a_1)\right)\right\rangle$ and $\tilde{a}_2 = \left\langle s_{\theta(a_2)}, \left(\mu(a_2), v(a_2)\right)\right\rangle$ are any two ILNs, then:

1) If $S\left(\tilde{a}_1\right) > S\left(\tilde{a}_2\right)$, then, $\tilde{a}_1 > \tilde{a}_2$;

2) If $S\left(\tilde{a}_1\right) = S\left(\tilde{a}_2\right)$), then

If $H\left(\tilde{a}_1\right) > H\left(\tilde{a}_2\right)$, then, $\tilde{a}_1 > \tilde{a}_2$;

If $H\left(\tilde{a}_1\right) = H\left(\tilde{a}_2\right)$, then, $\tilde{a}_1 = \tilde{a}_2$.

2.3 The IOWA operator

The IOWA operator was introduced by Yager and Filev (1999) and it represents an extension of the OWA operator. The main difference is that the reordering step of the IOWA is carried out with order-inducing variables, rather than depending on the values of the arguments. The IOWA operator also includes the maximum, the minimum and the average operators, as special cases. Since its appearance, the IOWA operator has been studied by different authors (Chen and Zhou, 2011; Merigó, 2011; Merigó and Gil-Lafuente, 2011c, 2013; Wei, 2010a; Xu and Wang, 2012; Yager et al., 2011; Zeng, 2013;). It can be defined as follows:

Definition 7. An IOWA operator of dimension n is a mapping IOWA: $R^n \times R^n \to R$ that has an associated weighting vector W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$IOWA\left(\langle u_1, a_1\rangle, ..., \langle u_n, a_n\rangle\right) = \sum_{j=1}^{n} w_j b_j \tag{7}$$

where b_j is a_i value of the IOWA pair $\langle u_i, a_i \rangle$ having the j th largest u_i, u_i is the order inducing variable and a_i is the argument variable.

2.4 The IOWAD operator

The IOWAD operator introduced by Merigó and Casanovas (2011a) is a distance measure that uses the IOWA operator in the normalization process of the Hamming distance. For two sets $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$, the IOWAD operator can be defined as follows:

Definition 8. An IOWAD operator of dimension n is a mapping IOWAD: $R^n \times R^n \times R^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$IOWAD\left((u_1, a_1, b_1), ..., (u_n, a_n, b_n)\right) = \sum_{j=1}^{n} w_j d_j \tag{8}$$

where d_j is the $|a_i - b_i|$ value of the IOWAD triplet (u_i, a_i, b_i) having the j th largest u_i, u_i is the order inducing variable and $|a_i - b_i|$ is the argument variable represented in the form of individual distances.

The main advantage of the IOWAD operator is that it is able to deal with complex attitudinal characters (or complex degrees of orness) in the decision process by using order-inducing variables. In doing so, we are able to deal with more complex problems that are closer to real-world situations. The IOWAD operator can be used in a wide range of different problems including decision-making, statistics, economics and engineering.

When using the IOWAD operator, it is assumed that the available information includes exact numbers or singletons. In order to extend the IOWAD operator to intuitionistic linguistic environment, in the following, we shall develop introduce the induced intuitionistic linguistic ordered weighted averaging distance (IILOWAD) operator.

3. The IILOWAD operator

The induced intuitionistic linguistic ordered weighted averaging distance (IILOWAD) operator is an extension of the IOWAD operator that uses uncertain information in the aggregation represented in the form of ILNs. It isan aggregation operator that unifies ILNs, the distance measures and the IOWA operator in the same formulation. Thus, we are able to provide a model that is able to assess the information in situations with high degree of uncertainty by using ILNs. Moreover, it provides a parameterized family of intuitionistic linguistic aggregation distance operators between the minimum operator and the maximum operator. Furthermore, it also considers complex reordering processes that permit to assess complex attitudinal characters of the decision maker. It can be defined as follows.

Definition 9. An IILOWAD operator of dimension n is a mapping IILOWAD: $\Omega^n \times \Omega^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$IILOWAD\left(\left\langle u_1, \tilde{a}_1, \tilde{b}_1 \right\rangle, \ldots, \left\langle u_n, \tilde{a}_n, \tilde{b}_n \right\rangle\right) = \sum_{j=1}^{n} w_j d_{ILNHD}\left(\tilde{a}_j, \tilde{b}_j\right) \qquad (9)$$

where Ω is the set of all ILN, $d_{ILNHD}\left(\tilde{a}_j, \tilde{b}_j\right)$ is $d_{ILNHD}\left(\tilde{a}_i, \tilde{b}_i\right)$ value of the IILOWAD pair $\left\langle u_i, \tilde{a}_i, \tilde{b}_i \right\rangle$ having the j th largest u_i, u_i is the order-inducing variable and $d_{ILNHD}\left(\tilde{a}_i, \tilde{b}_i\right)$ is the argument variable represented in the form of an individual distances.

In the following example, we present a simple numerical example showing how to use the IILOWAD operator in an aggregation process.

Example 1.

Let $A = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4) = \{\langle s_6, (0.5, 0.4)\rangle, \langle s_4, (0.3, 0.4)\rangle, \langle s_4, (0.6, 0.3)\rangle, \langle s_3, (0.2, 0.6)\rangle\}$ and $B = \left(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4\right) = \{\langle s_2, (0.3, 0.6)\rangle, \langle s_4, (0.5, 0.4)\rangle, \langle s_6, (0.7, 0.2)\rangle, \langle s_5, (0.5, 0.5)\rangle\}$ be two sets of all ILNs defined in a seven linguistic terms set $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ .Based on the Eq. (3), we can calculate the distance between the ILNs $\tilde{a}_1$ and $\tilde{b}_1$:

$$d_{ILHD}(\tilde{a}_1, \tilde{b}_1) = \frac{1}{2 \times (7-1)} \times \left(\left|(1+0.5-0.4)) \times 6 - (1+0.3-0.6) \times 2\right|\right) = 0.433$$

Similarly, we have

$$d_{ILHD}(\tilde{a}_2, \tilde{b}_2) = 0.067,\ d_{ILHD}(\tilde{a}_3, \tilde{b}_3) = 0.283,\ d_{ILHD}(\tilde{a}_4, \tilde{b}_4) = 0.267,$$

Assume the order-inducing variables $(u_1, u_2, u_3, u_4) = (2, 7, 4, 9)$, it yields:

$\left\langle u_1, \tilde{a}_1, \tilde{b}_1 \right\rangle = \left\langle u_1, d_{ILHD}(\tilde{a}_1, \tilde{b}_1) \right\rangle = \langle 2, 0.433 \rangle$, $\left\langle u_2, \tilde{a}_2, \tilde{b}_2 \right\rangle = \left\langle u_2, d_{ILHD}(\tilde{a}_2, \tilde{b}_2) \right\rangle = \langle 7, 0.067 \rangle$,
$\left\langle u_3, \tilde{a}_3, \tilde{b}_3 \right\rangle = \left\langle u_3, d_{ILHD}(\tilde{a}_3, \tilde{b}_3) \right\rangle = \langle 4, 0.283 \rangle$, $\left\langle u_4, \tilde{a}_4, \tilde{b}_4 \right\rangle = \left\langle u_4, d_{ILHD}(\tilde{a}_4, \tilde{b}_4) \right\rangle = \langle 9, 0.267 \rangle$,

If the weighting vector is $W = (0.4, 0.3, 0.1, 0.2)$, based upon the ordering-inducing variable u_i, we get an aggregated distance between A and B using the IILOWAD operator:

$$IILOWAD(A, B) = 0.4 \times 0.267 + 0.3 \times 0.067 + 0.1 \times 0.283 + 0.2 \times 0.433 = 0.242$$

From a generalized perspective of the reordering step, we can distinguish between the descending IILOWAD (DIILOWAD) operator and the ascending

IILOWAD (AIILOWAD) operator by using $w_j = w^*_{n-j+1}$, where w_j is the j th weight of the DIILOWAD and w^*_{n-j+1} the j th weight of the AIILOWAD operator.

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the IILOWAD operator can be expressed as:

$$IILOWAD\left(\left\langle u_1, \tilde{a}_1, \tilde{b}_1 \right\rangle, ..., \left\langle u_n, \tilde{a}_n, \tilde{b}_n \right\rangle\right) = \frac{1}{W} \sum_{j=1}^{n} w_j d_{ILNHD}\left(\tilde{a}_j, \tilde{b}_j\right) \quad (10)$$

Similar to the IOWAD operator, the IILOWAD operator is commutative, monotonic, bounded, idempotent, nonnegative and reflexive. Note that these properties can be proved with a similar method than the IOWAD operator and thus omitted.

An interesting issue is to consider the measures for characterizing the weighting vector W of the IILOWAD operator such as the attitudinal character, the entropy of dispersion, the divergence of W and the balance operator. As this feature does not depend upon the linguistic arguments, the formulation is the same than the IOWAD operator. The entropy of dispersion is defined as follows:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \quad (11)$$

This can be used to measure the quantity of information used. For example, if $w_j = 1$ for some j, then $H(W) = 0$, and thus the least amount of information is used.

The balance operator can be defined as:

$$Bal(W) = \sum_{j=1}^{n} w_j \left(\frac{n+1-2j}{n-1}\right) \quad (12)$$

It can be shown that $Bal(W) \in [-1,1]$. Note that for the optimistic criteria, $Bal(W) = 1$, and for the pessimistic criteria, $Bal(W) = -1$.

The divergence of W measures the divergence of the weights against the attitudinal character measure:

$$Div(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2 \quad (13)$$

The degree of orness can be defined as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right) \quad (14)$$

It can be shown that $\alpha \in [0,1]$. The more weight is located near the top of W, the closer α is to 1, while the more weight is located toward the bottom of W, the closer α is to 0.

A further issue is the problem of ties in the reordering process of the order-inducing variables. To solve this problem, we recommend the policy explained by Yager and Filev (1999), namely, replacing the tied arguments by their average. Note that in this case, it would mean replacing the tied arguments by their intuitionistic linguistic normalized Hamming distance.

4. Families of IILOWAD operators

By using a different manifestation of the weighting vector, we are able to obtain different types of IILOWAD operators, such as the intuitionistic linguistic normalized Hamming distance (ILNHD), the intuitionistic linguistic weighted Hamming distance (ILWHD), the intuitionistic linguistic ordered weighted averaging distance (ILOWAD) operator, the step-ILIOWAD, the window-IILOWAD, the median-IILOWAD, the Olympic-IILOWAD and the centered-IILOWAD.

Remark 1. For example, some of the most basic families such as the intuitionistic linguistic maximum distance (ILMAXD), the intuitionistic linguistic minimum distance (ILMIND), the step-IILOWAD, the ILNHD, the ILWHD and the ILOWAD are obtained as follows:

- The ILMAXD is obtained if $w_p = 1$, $w_j = 0$,for all $j \neq p$, and $u_p = \max\left\{d_{ILNHD}\left(\tilde{a}_i, \tilde{b}_i\right)\right\}$.
- The ILMIND is obtained if $w_p = 1$, $w_j = 0$,for all $j \neq p$, and $u_p = \min\left\{d_{ILNHD}\left(\tilde{a}_i, \tilde{b}_i\right)\right\}$.
- More generally, if $w_k = 1$ and $w_j = 0$ for all $j \neq k$, we get the step-IILOWAD operator.
- The ILNHD is formed when $w_j = 1/n$ for all j.
- The ILWHD is obtained when the ordered position of the u_i is the same as $d_{ILNHD}\left(\tilde{a}_i, \tilde{b}_i\right)$.
- The ILOWAD operator is obtained if the ordered position of u_i is the same as the ordered position $d_{ILNHD}\left(\tilde{a}_i, \tilde{b}_i\right)$.

Remark 2. Another particular case is the Olympic-IILOWAD. This operator is found when $w_1 = w_n = 0$ and for all others $w_{j*} = 1/(n-2)$. Note that if $n = 3$ or $n = 4$, the Olympic-IILOWAD is transformed in the median-IILOWAD.

Remark 3. Note that it is possible to present a general form of the Olympic-IILOWAD operator, considering that $w_j = 0$ for $j = 1, 2, ..., k, n, n-1, ..., n-k+1$, and for all others, $w_{j*} = 1/(n-2k)$ where $k < n/2$.

Note that if $k=1$, then this general form becomes the usual Olympic-IILOWAD. If $k=(n-1)/2$, then it becomes the median-IILOWAD operator.

Remark 4. Additionally, it is also possible to present the contrary case of the general Olympic-IILOWAD operator. In this case, $w_j = 1/(2k)$ for $j=1,2,...,k,n,n-1,...,n-k+1$, and $w_j=0$, for all others, where $k<n/2$. Note that if $k=1$, then we get the contrary case of the median-IILOWAD.

Remark 5. Note that the generalized median and the weighted generalized median can also be used as a particular case of the IILOWAD. For the IILOWAD median, if n is odd we assign $w_{(n+1)/2}=1$ and $w_j=0$ for all others, and this affects the argument $d_{ILNHD}\left(\tilde{a}_i,\tilde{b}_i\right)$ with the $[(n+1)/2]$th largest u_i. If n is even we assign for example, $w_{n/2}=w_{(n/2)+1}=0.5$, and this affects the arguments with the $[(n/2)]$th and $[(n/2)+1]$th largest u_i. For the weighted LIOWAD median, we select the argument $d_{ILNHD}\left(\tilde{a}_i,\tilde{b}_i\right)$ that has the kth largest inducing variable u_i, such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k-1$ is less than 0.5.

Remark 6. Another type of aggregation that could be used is the E-Z IILOWAD weights. In this case, we should distinguish between two classes. In the first class, we assign $w_j=(1/k)$ for $j=1$ to k and $w_j=0$ for $j>k$, and in the second class, we assign $w_j=0$ for $j=1$ to $n-k$ and $w_j=(1/k)$ for $j=n-k+1$ to n.

Remark 7. A further family that could be used is the centered-IILOWAD operator, which is based on Yager (2004). An IILOWAD operator can be defined as a centered aggregation operator if it is symmetric, strongly decaying and inclusive.

- It is symmetric if $w_j=w_{j+n-1}$;
- It is strongly decaying when $i<j\le(n+1)/2$ then $w_i<w_j$ and when $i>j\ge(n+1)/2$ then $w_i<w_j$;
- It is inclusive if $w_j>0$.

5. Decision making with the IILOWAD operator

The IILOWAD operator can be applied in a wide range of areas including statistics, economics and soft computing. In the following, we are going to present a numerical example of the new approach in a decision making problem about selection of public project. Note that other decision making applications could be developed such as in financial decision making (Xu and Hu, 2010; Ye, 2013), human resource management (Baležentis and Zeng, 2013) and engineering economic analysis decision (Kuchta, 2001; Omitaomu and Badiru, 2007).

Let consider a civil engineering public project selection problem. The same problem was also solved in reference Khademi et al. (2014) by using the AHP/ANP method. For simplicity, we assume that a government wants to select an optimal transportation system from three feasible alternatives $A_i (i=1,2,3)$: A_1 – conventional rail; A_2 – high speed rail; A_3 – Maglev.

The criteria for the selection of transportation system can be invented as follows: C_1 – cost of constructing the system (typically measured in dollars); C_2 – reliability of the system (typically measured as mean time to failure); C_3 – life cycle cost to the public (typically measured in monetary terms and includes the cost of ridership fares).

Due to the fact that the importance of each criterion is very imprecise because they contain a lot of particular aspects, the experts of government cannot use numerical values in the analysis. Instead, the importance of each criterion is evaluated using linguistic term set with seven terms: $S=\{s_0,s_1,s_2,s_3,s_4,s_5,s_6\}$ ={very unimportant, unimportant, slightly unimportance, middle, slightly important, important, very important }.

After careful analysis of these criterions, the experts give the evaluation information by the intuitionistic linguistic numbers using linguistic set $S=\{s_0,s_1,s_2,s_3,s_4,s_5,s_6\}$ shown in Table

1. For example, the ILN $\langle s_5,(0.5,0.4)\rangle$ in the Table 1, given by experts for alternative A_1 with respect to the criterion C_1, can be explained that the experts should give the membership degree (0.5) and non-membership degree (0.4) to their preference "important" (s_5) as they may not have enough expertise or possess a sufficient level of knowledge to precisely express their preferences over the objects. Therefore, it is formed the evaluation value ILN $\langle s_5,(0.5,0.4)\rangle$.

Table 1: The evaluation information given by experts

	C_1	C_2	C_3
A_1	$\langle s_5,(0.5,0.4)\rangle$	$\langle s_3,(0.3,0.4)\rangle$	$\langle s_4,(0.6,0.3)\rangle$
A_2	$\langle s_2,(0.3,0.6)\rangle$	$\langle s_4,(0.5,0.4)\rangle$	$\langle s_3,(0.7,0.2)\rangle$
A_3	$\langle s_4,(0.2,0.7)\rangle$	$\langle s_2,(0.6,0.2)\rangle$	$\langle s_5,(0.6,0.3)\rangle$

According to their objectives, the experts establish the following ideal alternative shown in Table 2. Note that this ideal information is based on the experts' achieving consensus agreement and negotiation.

Table 2: Ideal alternative

	C_1	C_2	C_3
Ideal	$\langle s_6,(0.8,0.1)\rangle$	$\langle s_6,(0.9,0.1)\rangle$	$\langle s_5,(0.9,0.1)\rangle$

To analyze the attitudinal character of the board of directors, we consider that they use order-inducing variables shown in Table 3, which represents the complex attitudinal character in the decision process. Note that in this example, the experts assume a different attitudinal character for each alternative because the results given by each alternative are not equal. The main advantage of using order inducing variables is that we can represent complex decision processes that include psychological factors such as time pressure, personal affects to each alternative and other related aspects.

Table 3: Order-inducing Variables

	C_1	C_2	C_3
A1	17	13	9
A2	12	6	20
A3	12	14	16

With this information, it is possible to develop different methods for selecting an investment. In this example, we will consider the ILMAX, ILMIND, the ILNHD, the ILWHD, the ILOWADandthe step-IILOWAD ($k=2$). Note that the weighting vector used is: $W=(0.2,0.5,0.3)$, i.e., the degree of orness $\alpha(W)=0.26$. With this information, it is possible to aggregate the available information in order to take a decision. The method consists in comparing the available investments with the ideal one by using the IILOWAD operator and its particular cases. The results are shown in Table 4.

Table 4. Aggregated Results

	ILMAXD	ILMIND	ILNHD	ILWHD	ILOWAD	IILOWAD	step($k=2$)
A1	0.675	0.317	0.486	0.526	0.463	0.526	0.467
A2	0.733	0.375	0.547	0.526	0.526	0.602	0.733
A3	0.683	0.208	0.519	0.533	0.586	0.581	0.667

As we can see, depending on the distance aggregation operator used, the optimal choice is different. Note that the lowest value in each method is the optimal result.

If we establish an ordering of the investments, a typical situation if we want to consider more than one alternative, we will get the following orders shown in Table 5. Note that the first alternative in each ranking is the optimal choice.

Table 5. Ranking of the Investment Strategies

	Ordering		Ordering
ILMAXD	$A_1 \succ A_3 \succ A_2$	ILOWAD	$A_1 \succ A_2 \succ A_3$
ILMIND	$A_3 \succ A_1 \succ A_2$	IILOWAD	$A_1 \succ A_3 \succ A_2$
ILNHD	$A_1 \succ A_3 \succ A_2$	step($k = 2$)	$A_1 \succ A_3 \succ A_2$
ILWHD	$A_1 = A_1 \succ A_3$		

As we can see, depending on the particular type of IILOWAD operator used, the results may be different. Note that in this problem the, ILMAXD is the most pessimistic aggregation because it considers only the highest distance, that is, the worst characteristic of an alternative. On the other hand, the ILMIND is the most optimistic one. The ILNHD is a neutral aggregation because it gives the same weights to all the characteristics. The ILWHD considers the weights of the characteristics. The ILOWAD operator assumes that we are in an uncertain environment where we can only aggregate the information considering the attitudinal character of the decision maker. The Olympic-IILOWAD doesn't aggregate the highest and lowest distances.The IILOWAD considers complex attitudinal characters of the decision-maker by using order-inducing variables.

6. Conclusions

The traditional IOWAD operator is generally suitable for aggregating information taking the form of real numbers, but it will fail when dealing with intuitionistic linguistic numbers. In this paper, we have presented a new operator called the IILOWAD operator. We analyzed it as an extension of the IOWAD operator that uses uncertain information represented in the form of intuitionistic linguistic numbers. We have studied some of its main properties and we have seen that it includes many different types of intuitionistic linguistic aggregation distance operators, such as the ILNHD, the ILWHD, the step-IILOWAD and the Olympic-IILOWAD operators.

We have developed an application of the new operator in a decision making problem about selection of civil engineering public project. We have seen that the main advantage of this approach is that we can consider a wide range of future scenarios according to our interests and select the one that it is closest to our real interests.

In future research, we expect to present further extensions to this approach by using other distance measures such as the Euclidean distance, the Minkowski distance and the quasi-arithmetic distance. Moreover, other characteristics will be used such as probabilistic information and more general formulations by using weighted averages. Furthermore, other potential problems in other areas will be studied such as in financial and production management.

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