LINEAR PROGRAMMING METHODS FOR SOLVING THE PORTFOLIO'S PROBLEMS

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Abstract

It is well known that the portfolio optimization involves creating the stock portfolio minimizing the risk for a required return or maximizing the return for a given risk level. The mathematic model of these kind of problem is one of quadratic programming type. Because the solving procedure of these type of models is more complicated, in the proposed work will bring alternative models for solving a portfolio's problem. Particularly in the paper is proposed some techniques and considerations for non-linear portfolio's model transformation in one of linear or linear fractional type. The last ones leads to streamline the process of solving the initial model. The proposed methods have been verified practically on several examples and have been found very effective.

Key words: portfolio, risk, benefit, linear programming, fractional programming.

JEL Classification: G11, C61

1. Introduction

The portfolio problem consists in the determination of some efficient portfolios that can be proposed to the economic agents that plan an investment in valuable papers, to obtain a maximal income at a minimal risk. Professor Harry M. Markowitz [6] said, "a good portfolio is more than a long list of securities. It is a balanced unity which similarly offers the investor chance and protection below a variety of possible future developments. The investor should therefore aim at an integrated portfolio which takes his individual requirements into account." He won the Nobel Prize in 1990 for his research in the field of economic portfolio theory. His Portfolio Theory is based on empirical sizes, which analyze the connection between risk and return. Every economic agent knows that for to obtain a higher benefit, it is supposed to take a certain risk. *The portfolio means a set of real assets, including financial, in which an Enterprise can have its investments at a given moment.* [1],[2]. In portfolio's decision, the Enterprise should begin with the fact that the structure and characteristics of the entire portfolio are more important and require more attention than each asset of the portfolio by part; an extremely risky asset thus can be kept in a portfolio together with other assets with a lower degree of risk. When the decision to add an asset to an existing portfolio is made, the Enterprise should consider the effect that it will have on the value of the Enterprise, but also on the structure of its entire portfolio. For analyzing this last influence, the deciding part should consider *the rate of the income of portfolio and the risk,* represented the average and standard deviations. The portfolio theory supposes that for a certain risk level, economic agents plan an investment of a certain amount in securities, the purchase proportion being elected with the intention to have a maximal benefit at a minimal risk. The benefit shall be measured as the average value between continuously calculated rates during the year, while risk – in terms of dispersion and covariance. It is obvious that for economic agents it is more suitable to make investments in an efficient portfolio. *A securities portfolio is considered efficient if there is no other portfolio with higher benefit and lowed degree of risk, with a higher benefit and the same risk degree or with the same benefit and a lower degree of risk.* Nowadays problems of optimal portfolios acquired a very large scale, because it came in various economic and financial sectors. Thus the optimal portfolio problems are studied in various forms by deterministic or fuzzy data [3]. Frequently it is formulated as a multiple criteria optimization problem for which use special algorithms [7], leading to the determination of a set of solutions effectively. As a rule, but the decision maker selects between them a solution, the best, called optimal, which increases the importance of one criterion optimization models.

2. Mathematical optimization model of the portfolio problem

If we understand the concept of the efficiency of portfolio we can speak about the formulation of the problem. We have two tendencies in the portfolio problem: minimizing risk and maximizing the benefit. We will study several models of quadratic programming, which once being solved will

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determinate efficient portfolios. They can be founded in dependence of the requests imposed by the investor [6].

We make the next notations:

x_j – the part of the capital reserved for purchasing securities of j type,

r_j – the average value of the efficiency of j type securities during the whole T period,

$\sigma_{ij} = \frac{1}{T}\sum_{t=1}^{T}(x_{it} - r_i)(x_{jt} - r_j)$ – the covariance of securities,

x_{jt} – dobanda obtained from the investment in j – type securities during t period,

B – the total budget invested in the portfolio,

i, j – types of securities,

T – the whole investment period,

t – a period of time taken from T with a certain step,

q – the maximal level of risk that the investor is able to take.

We find the next optimization model:

$$\sum_{j=1}^{n} r_j x_j \to \max,$$

$$\sum_{j=1}^{n} \sigma_{ij} x_i x_j \leq q, \qquad (1)$$

$$\sum_{j=1}^{n} x_j = B = 1, \qquad x_j \geq 0.$$

In this case, the investor is interested in maximizing the benefit by assuming a certain risk, which level doesn't have to pass the q value.

When the economic agent that plans the investment tends to a minimization of risk while having a pre-established benefit, the mathematical model is modified, taking in consideration the new requests:

$$\sum_{j=1}^{n} \sigma_{ij} x_i x_j \to \min,$$

$$\sum_{j=1}^{n} r_j x_j \geq \propto \cdot B \qquad (2)$$

$$\sum_{j=1}^{n} x_j = 1, \qquad x_j \geq 0.$$

$\propto$ – the minimal benefit requested by the investor.

The portfolio problem raises a special interest when two objectives that require simultaneous optimization are imposed. In this case, an objective – function of aggregation is examined, and it is represented as follows:

$$z = \sum_{j=1}^{n} r_j x_j - \sum_{j=1}^{n} \sigma_{ij} x_i x_j \rightarrow \max \qquad (3)$$

$$\sum_{j=1}^{n} x_j = 1, \;\; x_j \geq 0.$$

So but optimal portfolio problem solving assumes either solving the problem of non-linear multi-criteria optimization, ie models 1-2 or solving the problem 3, which is also non-linear. In the first case is going to propose efficient algorithms for constructing a set of efficient solutions, while in the second only an optimal solution.

Definition 1. The vector of invest $X = (X_1, X_2, ..., X_n)$ is one available portfolio for the portfolio's model if it satisfies the availability conditions of this.

Definition 2. The available portfolio $X^* = (X_1^*, X_2^*, ..., X_n^*)$ is one optimal if it optimizes the objective function of portfolio's model.

This type of model that targets are the simultaneous optimization of two objectives was proposed by Markowitz that - as previously been mentioned - for the first time in the modern history of portfolio has described the risk - yield relation and constructed the famous theory of *Average-Dispersion* [6]. Based on his theory, investors can find optimal portfolios. This conclusion was a very important step for the theory of capital markets. William Sharpe said about Markowitz that he came with an idea that made order and light in the way where the investors had to choose their titles.

Some inconveniences of the Markowitz's model:

•*Computational burden: We have to calculate n(n+1)/2 constants of deviations; it's not easy to obtain an optimal solution of large- scale quadratic programming problems on a real time basis.*

•*Investors' perception against risk and distributions of stock prices: Many practitioners were not fully convinced of the validity of the standard deviation as a measure of the risk; the investors' perception against risk is not symmetric around the mean. This is why the Markowitz's model should be viewed as an approximation to the more complicated optimization problem facing an investor.*

•*Transaction/ Management cost and cut-off effect: An optimal solution of a large-scale quadratic programming problem many nonzero element, that is very inconvenient from practical point of view, because we have to pay significant amount of transaction costs to buy many different stocks by a small amount. We may not be able to purchase small amount. We may not be able to purchase small amounts of stock below minimum transaction unit. We have to round the numbers to the integer multiples and solve the integer quadratic programming problem.*

3. Model with linear fractional criteria Sharpe [8]

The main idea of the method consists of replacing of the two criteria Markowitz portfolio's model (1)- (2), (3) with one of the linear fractional type.

Keeping the previous notations and accepting these:

c $R_j - a\ random\ \mathrm{var}iable\ of\ return\ rata\ for\ asset - j,$

$r_j - the\ \exp ected\ return\ (per\ \ period),\ often\ given\ by\ relation\ r_j = E[R_j],$

$\sigma_{ij} = E[(R_i - r_i)(R_j - r_j)] - the\ \mathrm{cov}\, ariance\ of\ \sec urities,$

$\sigma(x_1, x_2, ...x_n) = \sqrt{E\left[\left\{\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right\}^2\right]} - s\tan dard\ diviation;$

We propose to study the optimal portfolio model in which the objective function is of fractional type, of course keeping the conditions of admissibility of the solution, it is the next:

$$\frac{\sum_{j=1}^{n} r_j x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i \sigma_{ij} x_j}} \to \max \quad (4)$$

The proposed criterion involves maximizing the return on the portfolio per unit of risk.

Is also popular, another, first proposed by Sharpe criterion:

$$\frac{\sum_{j=1}^{n} r_j x_j}{\sum_{j=1}^{n} \beta_j x_j} \to \max \quad (5)$$

where β_j - regression coefficient between stock returns and market *j* .

The solution procedure of the problem (4) - (5) is reduced to the applying of greedy algorithm in the sequence for a simple linear problems with parameter [4].

4. Linearization procedure of the portfolio's model

We introduce the next risk function [5] of absolute deviation:

$$\omega(x) = E\left[\left|\sum_{j=1}^{n} R_j x_j - E\left[\sum_{j=1}^{n} R_j x_j\right]\right|\right]$$

Instead of the standard deviation function per each time period.

$$(\sigma(x) = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j} - the\ s\tan dard\ deviations)$$

$$(R_1, R_2, \ldots R_n)$$

These two measures are essentially the same if are multivariate normally distributed.

Theorem

$$If\ (R_1, R_2, \ldots R_n)\ are\ multi\operatorname{var}iate\ normally\ distributed, then\ the\ relation:$$

$$\omega(x) = \sqrt{\frac{2}{\pi}}\sigma(x)\ is\ true.$$

Proof

Let: $(\mu_1,\mu_2,...\mu_n)$ *be the mean of* $(R_1,R_2,...R_n)$ *and* $(\sigma_{ij})\in R^{n\times n}$ *be the covariance matrix of* $(R_1,R_2,...R_n)$, *then* $\sum_{j=1}^{n} R_j x_j$ *is normally distributed* (*Rao*, 1965) *with mean* $\sum_{j=1}^{n}\mu_j x_j$ *and standard deviation* $\sigma(x)=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}$.

Therefore,

$$\omega(x)=\frac{1}{\sqrt{2\pi}}\times\frac{1}{\sigma(x)}\int_{-\infty}^{+\infty}|u|\exp\left(-\frac{u^2}{2\sigma^2(x)}\right)du=\sqrt{\frac{2}{\pi}}\sigma(x).$$

The last relation proof the linearity of the risk function of absolute deviation and implies that minimizing $\omega(x)$ is equivalent to minimizing $\sigma(x)$ if $(R_1,R_2,...,R_n)$ is multivariate normally distributed. Thus we are led to an alternative risk minimizing problem, which is the following:

$$\min E\left[\left|\sum_{j=1}^{n}R_j x_j-E\left[\sum_{j=1}^{n}R_j x_j\right]\right|\right]$$

$$\text{subject to } \sum_{j=1}^{n}E[R_j]x_j\geq\alpha B, \qquad (6)$$

$$\sum_{j=1}^{n}x_j=B,$$

$$0\leq x_j\leq u_j,\ j=1,2,...n.$$

In model (8) we kept the notations and meanings of variables like from the models (1)-(3) and also considering that the variables $\{x_j\}$ have certain limits of variation $\{u_j\}$, $j=1,2,...n$

Either we suppose: r_{jt} be the realization of random of variable R_j during period $t(t=1,2,...T)$, and that the expected value of the random variable can be approximate by the average derived from certain data. In particular we can consider: $r_j=E[R_j]=\sum_{t=1}^{T}r_{jt}/T$.

Then $\omega(x)$ can be approximated as follows:

$$E\left[\left|\sum_{j=1}^{n}R_j x_j-E\left[\sum_{j=1}^{n}R_j x_j\right]\right|\right]=\frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j=1}^{n}(r_{jt}-r_j)x_j\right|$$

By noting with: $a_{jt}=r_{jt}-r_j$, $j=1,2,...,n;$ $t=1,2,...,T.$, the model (6) becomes the next:

$$\min\sum_{t=1}^{T}\left|\sum_{j=1}^{n}a_{jt}x_j\right|/T$$

$$\text{subject to } \sum_{j=1}^{n}r_j x_j\geq\alpha B, \qquad (7)$$

$$\sum_{j=1}^{n}x_j=B,$$

$$0\leq x_j\leq u_j,\ j=1,2,...n.$$

By applying of some elementary transformations in the model (7), we obtain the its equivalent model (8), which is the following:

$$\min \sum_{t=1}^{T} y_i / T$$

$$\begin{aligned} \textit{subject to} \quad & y_i + \sum_{j=1}^{n} a_{ij} x_j \geq 0,\ t = 1,2,....T, \\ & y_i - \sum_{j=1}^{n} a_{ij} x_j \geq 0,\ t = 1,2,...T, \\ & \sum_{j=1}^{n} r_j x_j \geq \alpha B, \\ & \sum_{j=1}^{n} x_j = B, \\ & 0 \leq x_j \leq u_j,\ j = 1,2,...n \end{aligned} \qquad (8)$$

We can state, that the optimization model (8) is one of linear type. Obviously, comparing the obtained model (8) with the Markowitz 's model (3) and with Sharpe's models with objectives (4) or (5), we can say, that the latter certainly is more simply to solve both in terms of methodology, and of the solving time. In order to solve the model (8), we can use quite effectively simplex algorithm for solving of linear programming problems.

5. Conclusions

Portfolio problem remains one of the most requested in solving decisional situations not only in the financial sector, but also from various economic fields. Due to the nature of the portfolio problem and its formulation, and the requirement to find at least one its optimal solution, confirm evidently of the nature of the mathematical optimization problem. In this paper we focused on simplifying its procedures, reducing it to one linear programming type. This was possible using the absolute deviation for the risk function of the classical Markowitz's portfolio model. So, we made it possible to use the classical simplex algorithm for solving the eventually a nonlinear programming problems. The necessary theoretical justifications are made in the paper. The method we have developed is accurate and efficient enough to solve realistic problems in a reasonable amount of time.

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