12. On Tobin's Multiperiod Portfolio Theorem[1]

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Abstract

This paper investigates the multi-period portfolio problem under the framework of Tobin. Specifically, the paper analyzes the optimal two-period portfolio strategy compared with the buy-and-hold strategy, the stochastic rebalancing strategy and the simple rebalancing strategy. According to the result of the numerical example, both the non-Tobin strategy and stochastic rebalancing strategy are better than Tobin strategy, even near the origin. Therefore, the Tobin's multiperiod portfolio theorem is not always true.

Keyword: Tobin's multiperiod portfolio Theorem, the simple rebalancing Strategy (Tobin's Strategy), non-Tobin's Strategy, the buy-and-hold Strategy, the stochastic rebalancing strategy

JEL Classification: G11

1. Introduction

The analysis of the discrete multi-period portfolio problem can date back to the studies in 1960's, including Tobin(1965), Mossin(1968), Fama(1970), Hakansson(1974), Chen, Jan and Zionts(1971), Elton and Gruber(1971), etc. Tobin(1965) first presents a simple framework, based on which he gives the optimal multi-period portfolio theorem. But Stevens(1971) finds counterexamples to Tobin's theorem. Recently, Duan Li(2000) gives an innovative optimal discrete multi-period portfolio theorem which is inconsistent with Tobin's. In this paper, we develop a further analysis of Tobin theorem's faults based on the Steven's counterexamples, and compare it with Duan Li's theorem.

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2. Tobin's multiperiod portfolio theorem

Tobin(1965) assumed that the investment horizon is *n* periods of equal length and the stochastic price-change generating process is stationary and that successive price changes are mutually,independent,and then the asset returns are independent as among periods.Following Tobin,let the return ,expectation ,and risk of the portfolio planned for the ith period be R_i , E_i ,and σ_i . i.e. $E_i = E(R_i)$, $\sigma_i = \sqrt{Var(R_i)}$ .Let the over-all *n*-period return, expectation, and risk be *R, E, and* σ .By definition $1+R=\prod_{i=1}^{n}(1+R_i)$ .And by assumption of independence, we have

$$E(1+R)\equiv 1+E=\prod_{i=1}^{n}(1+E_i) \tag{1}$$

$$\sigma^2 \equiv Var(1+R)=E(1+R)^2-(E(1+R))^2 \tag{2a}$$

Then

$$\sigma^2+(1+E)^2=E(1+R)^2=\prod_{i=1}^{n}[\sigma_i^2+(1+E_i)^2] \tag{2b}$$

Tobin pointed out that:

'Within any period the variance of return σ_i dependens on the expectation of return R_i in the locus of efficient risk-expectation opportunities. The assumption of stationarity means that this relationship is the same in every period: $\sigma_i^2 = f(1+E_i)$ '. Then,the multi-period portfolio selection problem is:

$$\min \prod_{i=1}^{n}[\sigma_i^2+(1+E_i)^2]$$

$$subject \quad to \quad \prod_{i=1}^{n}(1+E_i)=1+E$$

By Lagrange multiplier Tobin conclued **the multiperiod portfolio theorem**:

'This can only be true for $1+E_1=1+E_2=...=1+E_n$ .Thus the conditions of the constrained extremum are met only by equalizing expectations,and with them risks,in all periods' .This means' the investor's optimal sequence of portfolios through time would be a stationary sequence-a series of portfolios with constant proportionate holdings of each included asset and ,consequently,a constant expected return and risk per dollar of invested wealth.'

TheTobin's multiperiod portfolio theorem indicates that investors only need to determine the single-period portfolio at the beginning and rebalance it at the end of each single period to make the initial portfolio of each period is identical, which is known as the simple rebalancing strategy.

3. Steven's counterexamples

Stevens (1971) showed the Tobin theorem is not, in general, true by some counter-examples. He considered a two-period variant of the Tobin problem:

$$\min[\sigma_1^2 + (1+E_1)^2][\sigma_2^2 + (1+E_2)^2]$$
$$subject \quad to \quad (1+E_1)(1+E_2) = K^2$$

Let $\sigma_i^2 = f(1+E_i)$ have the following special form for both periods:

$$\sigma_i^2 = [(1+E_i) - (1+r^*)]^2 = (E_i - r^*)^2 \quad (3)$$

Where r^* can be thought of as the riskless rate of interest. The function was defined for $E_i \geq r^*$ .Under that condition and r^*= 0 and K^2 = 4 the solution for the Tobin's problem is:

(a) $1+E_1 = 1+E_2 = 2$ (The simple rebalancing strategy)

(b) $1+E_1 = 1+E_2 = -2$ (The simple rebalancing strategy)

(c) $1+E_1 = \frac{9}{4} + \frac{\sqrt{68}}{8}$ $1+E_2 = \frac{9}{4} - \frac{\sqrt{68}}{8}$ (Not the simple rebalancing strategy)

(d) $1+E_1 = \frac{9}{4} - \frac{\sqrt{68}}{8}$ $1+E_2 = \frac{9}{4} + \frac{\sqrt{68}}{8}$ (Not the simple rebalancing strategy)

It is easy to find that (c) and (d) are better than (a) and (b), which indicates sometimes the simple rebalancing strategies are not the best. So sometimes the Tobin's theorem is wrong.

4.Rethingk the Stevens' Counter-example

To show that Tobin theorem does not always hold, Stevens gave three examples: $\sigma_i^2 = e^{AE_i} - (1+E_i)^2$, $\sigma_i^2 = (E_i - r^*)^2$ and $\sigma_i^2 = E_i^m$ (where m is any positive integer). However, we want to ask whether these functions are reasonable. For the purposes of the section, we will consider the simple two-period case. Using the notation introduced in last section, we want to solve the following problem:

$$\min \; [\sigma_1^2 + (1+E_1)^2][\sigma_2^2 + (1+E_2)^2]$$
$$s.t \;\; (1+E_1)(1+E_2) = 1+E$$

It is easy to prove that: if and only if the equations $\sigma_i^2 = f(1+E_i), (i=1,2)$ lie on the single-period efficient frontier the $(\sigma_1, E_1), (\sigma_2, E_2)$ solve the above problem.Huang and Litzenberger(1988) proved that the single-period efficient portfolio satisfies the following equation:

$$\begin{aligned}\sigma_i^2 &= \frac{C}{D}(E_i - \frac{A}{C})^2 + \frac{1}{C} \\ &= \frac{C}{D}(E_i + 1)^2 - \frac{2(A+C)}{D}(E_i + 1) + \frac{2A+B+C}{D}\end{aligned} \tag{4}$$

Where

V is the N×N variance-covariance matrix of the expected instantaneous rate of return of risky assets.

I is the N×1vector of 1, i.e $I' \equiv (1,1,...,1)$.

e is the N×1vector of expected instantaneous rate of return of risky assets,i.e .

$e' \equiv (E(R_1), E(R_2), ..., E(R_N))$, where R_i is the relative return for the *i*th risky asset.

N is the number of risky assets.

And define:

$$\begin{cases} A = I'V^{-1}e \\ B = e'V^{-1}e \\ C = I'V^{-1}I \\ D = BC - A^2 \end{cases} \tag{5}$$

Denote: $X_i \equiv 1 + E_i$, $\alpha = C/D, \beta = 2(A+C)/D, \gamma = (2A+B+C)/D$, rewrite (4) as:

$$\sigma_i^2 = \alpha(1+E_i)^2 - \beta(1+E_i) + \gamma \tag{6a}$$

$$\sigma_i^2 + (1+E_i)^2 = (1+\alpha)X_i^2 - \beta X_i + \gamma\ . \tag{6b}$$

Rewrite above problem as:

$$\begin{gathered}\min[(1+\alpha)X_1^2 - \beta X_1 + \gamma][(1+\alpha)X_2^2 - \beta X_2 + \gamma] \\ s.t \quad X_1 X_2 = 1 + E \equiv K^2\end{gathered}$$

Because

$$\begin{aligned}\sigma^2 + (1+E)^2 &= [(1+\alpha)X_1^2 - \beta X_1 + \gamma][(1+\alpha)X_2^2 - \beta X_2 + \gamma] \\ &= (1+\alpha)^2 K^4 + \beta^2 K^2 + (1+\alpha)\gamma(X_1^2 + X_2^2) - \beta[(1+\alpha)K^2 + \gamma](X_1 + X_2) + \gamma^2 \\ &= (1+\alpha)\gamma(X_1 + X_2)^2 - \beta[(1+\alpha)K^2 + \gamma](X_1 + X_2) + [(1+\alpha)K^2 - \gamma]^2 + \beta^2 K^2\end{aligned} \tag{7}$$

By Weda's Theorem, for a given two-period expectation of K^2, while

$$X_1 + X_2 = \frac{\beta[(1+\alpha)K^2 + \gamma]}{2(1+\alpha)\gamma} \tag{8}$$

We can get the minimum variance and

$$\sigma^2_{\min} = \frac{4(1+\alpha)\gamma\{[(1+\alpha)K^2-\gamma]^2+\beta^2K^2\}-\beta^2[(1+\alpha)K^2+\gamma]^2}{4(1+\alpha)\gamma} - K^4 \quad (9a)$$

Or

$$\sigma^2_{\min} = \frac{4(1+\alpha)\gamma\{[(1+\alpha)(1+E)-\gamma]^2+\beta^2(1+E)\}+\beta^2[(1+\alpha)(1+E)+\gamma]^2}{4(1+\alpha)\gamma} - (1+E)^2 \quad (9b)$$

where (9a) or (9b) is the two-period efficient frontier. Obviously, the two-period efficient frontier is still hyperbolic. Then, Tobin's theorem need not hold due to equation (8).The portfolio after rebalancing at the end of the first period is not equivalent to the initial one. We call that strategy as the non-Tobin's strategy.

By Tobin's theorem, we can derive the two-period efficient frontier for Tobin's strategy (or the simple rebalancing strategy. Tobin's strategy indicates $X_1 = X_2 = (1+E_1) = (1+E_2)$ .Then $X_1X_2 = (1+E_1)(1+E_2) = X^2$,and the two-period efficient frontier for Tobin's strategy is:

$$\begin{aligned}\sigma^2 &= [(1+\alpha)X_1^2 - \beta X_1 + \gamma][(1+\alpha)X_2^2 - \beta X_2 + \gamma] - (1+E)^2 \\ &= [(1+\alpha)X^2 - \beta X + \gamma]^2 - X^4 \\ &= [(2+\alpha)(1+E) - \beta\sqrt{1+E} + \gamma][\alpha(1+E) - \beta\sqrt{1+E} + \gamma]\end{aligned} \quad (10)$$

Furthermore, with the Tobin's assumptions in last section, Lee, Wu and Wei(1990) proved

$$1+\mathrm{E}_i(2)=[1+\mathrm{E}_i(1)]^2 \quad (11a)$$

$$\sigma_i^2(2)=[(1+\mathrm{E}_i(1))^2+\sigma_i^2(1)]^2-(1+\mathrm{E}_i(1))^4 \quad (11b)$$

$$\sigma_{ij}(2)=[(1+\mathrm{E}_i(1))(1+\mathrm{E}_j(1))+\sigma_{ij}(1)]-(1+\mathrm{E}_i(1))^2(1+\mathrm{E}_j(1))^2 \quad (11c)$$

Where

$\mathrm{E}_i(k)(k=1,2)$ is the expectation of the *k*-period return relative of the *i*th security

$\sigma_i^2(k)(k=1,2)$ is the variance of the *k* -period return relative of the *i*th security

$\sigma_{ij}(k)(k=1,2)$ is the *k* -period covariance between the return relative of the securities *i* and *j*

Accordingly, the variance-covariance matrix for buy-and-hold strategy is

$$V(2) \equiv \begin{bmatrix} \sigma_1^2(2) & \sigma_{12}(2) & \dots & \sigma_{1N}(2) \\ \sigma_{12}(2) & \sigma_2^2(2) & \dots & \sigma_{2N}(2) \\ \dots & \dots & \dots & \dots \\ \sigma_{1N}(2) & \sigma_{2N}(2) & \dots & \sigma_N^2(2) \end{bmatrix} \quad (12)$$

And the efficient frontier for buy-and hold strategy is:

$$\sigma^2(2) = \alpha(2)(1+E(2))^2 - \beta(2)(1+E(2)) + \gamma(2) \tag{13}$$

Where

$$\alpha(2) \equiv C(2)/D(2), \beta(2) \equiv 2[A(2)+C(2)]/D(2), \gamma(2) \equiv [2A(2)+B(2)+C(2)]/D(2)$$

$$\begin{cases} A(2) = I'V(2)^{-1}e(2) \\ B(2) = e'(2)V(2)^{-1}e(2) \\ C(2) = I'V(2)^{-1}I \\ D(2) = B(2)C(2) - A(2)^2 \end{cases} \tag{14}$$

Different from the approach above, Duan & Wan-Lung (2000) got a stochastic rebalancing strategy in the same mean-variance framework (see page 390-392). According to their definition we can get the notations under their 2-periods: $E(P') \equiv [E(R_2 - R_1), E(R_3 - R_1), ..., E(R_N - R_1)]$ and under the stationary assumption, $E(P_t'), E(P_t P_t'), E(R_1 P_t)$ do not vary with time changes. So, the mean-variance efficient frontier for Li's is:

$$\sigma^2 = \frac{a}{v^2}[E+1-(\mu+bv)]^2 + c \tag{15}$$

Where

$$\begin{cases} a = v/2 - v^2, b = \mu v/a, c = \tau - \mu^2 - ab^2 \\ \mu = J^2, v = JK + K, \tau = L^2 \\ J = E(R_1) - E(P')E^{-1}(PP')E(R_1 P) \\ K = E(P')E^{-1}(PP')E(P) \\ L = E[(R_1)^2] - E(R_1 P')E^{-1}(PP')E(R_1 P) \end{cases} \tag{16}$$

(See Duan & Wan-Lung(2000) page 391)

5. A numerical example

Following Duan & Wan-Lung (2000), consider the case in Chapter 7 of Sharpe, Alexander and Bailey (1995) by assuming a stationary 2-period process. There are three risky assets. The expected returns of risky assets A, B, and C are $1+E_A=E(R_A)=1.162$, $1+E_B=E(R_B)=1.246$, $1+E_C=E(R_C)=1.228$. The vector of the single-period expected yields is as follows:

$e' = (16.2\%,\ 24.6\%,\ 22.8\%)$. The covariance matrix of e' is:

$$V = \begin{bmatrix} 1.46\% & 1.87\% & 1.45\% \\ 1.87\% & 8.54\% & 1.04\% \\ 1.45\% & 1.04\% & 2.89\% \end{bmatrix}$$

By (5), we have: $A = 10.7557$, $B = 2.2106$, $C = 69.8459$, $D = 38.7199$,and $\alpha = 1.8039$, $\beta = 4.1633$, $\gamma = 2.4165$ .Then by (9b), the two-period efficient frontier for non-Tobin's strategy is:

$$\sigma^2_{\min} = 1.8039(1+E)^2 - 4.1633(1+E) + 2.4165 \tag{17}$$

And by (10), the two-period efficient frontier for Tobin's strategy is:

$$\begin{aligned}\sigma^2 &= 6.8617(1+E)^2 - 23.3468(1+E)^{3/2} + 30.8845(1+E) \\ &+20.1216(1+E)^{1/2} + 5.8396\end{aligned} \tag{18}$$

By (11)-(14), we have $1\text{+E}_\text{A}(2)\text{=}1.162^2 = 1.3502$, $1\text{+E}_\text{B}(2)\text{=}1.246^2 = 1.5525$ and $1\text{+E}_\text{C}(2)\text{=}1.228^2 = 1.5080$

$$V(2) = \begin{bmatrix} \sigma^2_A(2) & \sigma^2_{AB}(2) & \sigma^2_{AC}(2) \\ \sigma^2_{AB}(2) & \sigma^2_B(2) & \sigma^2_{BC}(2) \\ \sigma^2_{AC}(2) & \sigma^2_{BC}(2) & \sigma^2_C(2) \end{bmatrix} = \begin{bmatrix} 0.0396 & -0.6297 & -0.5947 \\ -0.6297 & 0.2725 & -0.8007 \\ -0.5947 & -0.8007 & 0.0880 \end{bmatrix}$$

And the two-period efficient frontier for buy-and-hold strategy is:

$$\sigma^2(2) = 168.6206(E+1)^2 - 409.0874(E+1) + 247.7126 \tag{19}$$

By (15) (16) and Duan & Wan-Lung's illustrative cases (see page 403), we have: $E(P) = (0.084, 0.066)'$, then:

$$\begin{cases} J = E(R_1) - E(P')E^{-1}(PP')E(R_1P) = 0.7424 \\ K = E(P')E^{-1}(PP')E(P) = 0.3566 \\ L = E[(R_1)^2] - E(R_1P')E^{-1}(PP')E(R_1P) = 0.8711 \\ \mu = J^2 = 0.5512, v = J^2K/L + K/2 = 0.2911, \tau = L^2 = 0.7588 \\ a = v/2 - v^2 = 0.0608, b = \mu v/a = 2.6385, c = \tau - \mu^2 - ab^2 = 0.0317 \end{cases}$$

The two-period efficient frontier for the Duan & Wan-Lung's stochastic rebalancing strategy is:

$$\sigma^2 = 0.7175E^2 - 0.4582E + 0.1048 \tag{20}$$

Figure 1

The mean-variance frontiers for all strategies

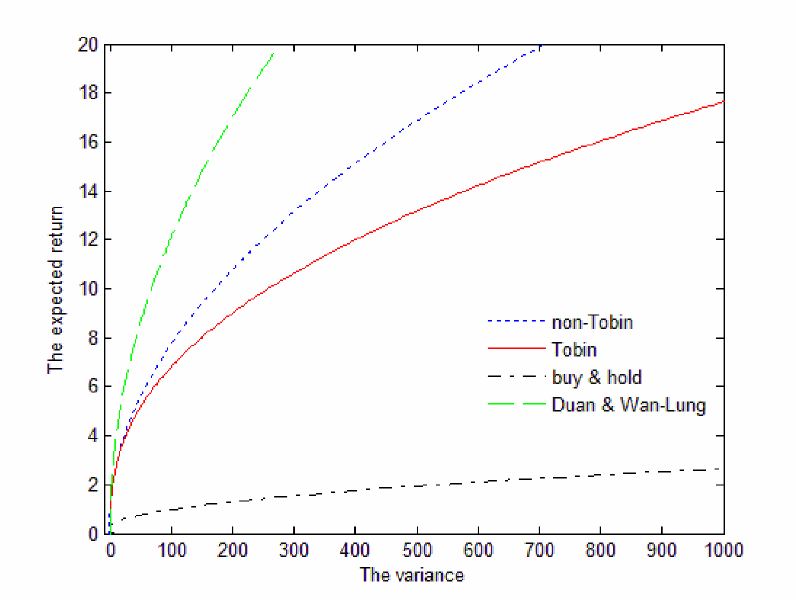

Figure 2

The mean-variance frontiers near the origin

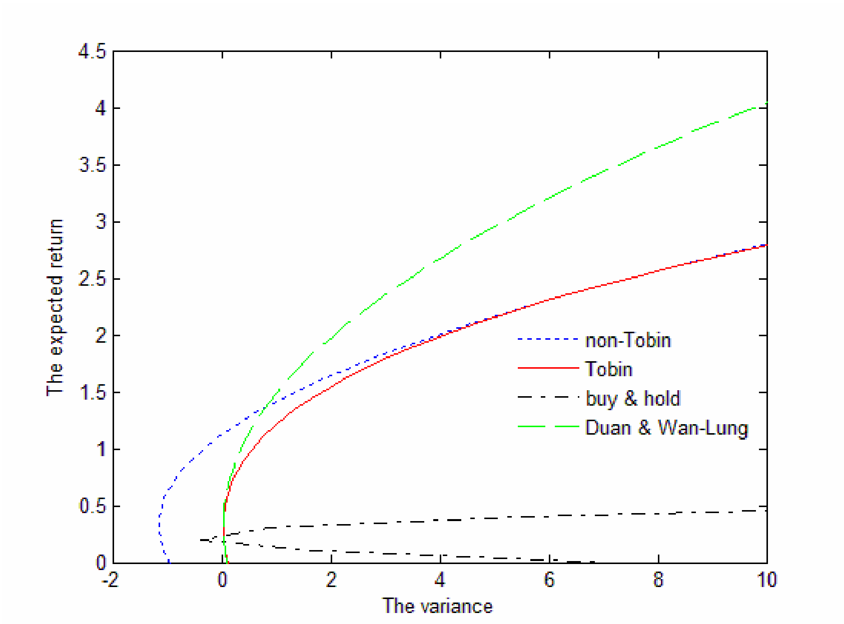

Figure 1 depicts the portfolio frontiers for non-Tobin strategy, Tobin strategy ,stochastic rebalancing strategy and buy-and-hold strategy .The time horizons are the same for these four strategies, but for first three strategies time horizon be divided into two equal length single-periods, and the last one only one-period. The figure 1 shows that with the same time horizion, the stochastic rebalancing strategy is the best one, and the buy-and-hold strategy is the worst one.non-Tobin strategy is superior to Tobin strategy. Figure 2 depicts the portfolio frontiers near the origin. The figure shows both the non-Tobin strategy and stochastic rebalancing strategy are better than Tobin strategy. Therefore, in our numerical example the Tobin's *multiperiod portfolio theorem* is not true.

6. Conclusion

In this paper, we discuss the further study on the fault of Tobin's theorem based on Steven's analysis under the assumption of independent stationary distribution and compare the results of four strategies: the simple rebalancing strategy based on Tobin's theorem, the buy-and-hold strategy based on the single-period efficient frontier $\sigma_i^2 = f(1+E_i)$, the dynamic rebalancing strategy of Duan Li and non-Tobin's strategy. By the numerical example, we find that the stochastic rebalancing strategy is the best one, and the buy-and-hold strategy is the worst one. And non-Tobin strategy is superior to Tobin strategy. So far as, we find near the origin both the non-Tobin strategy and stochastic rebalancing strategy are better than Tobin strategy. Therefore, the Tobin's *multiperiod portfolio theorem* is not always true.

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