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AN ANALYTIC DERIVATION OF THE EFFICIENT MARKET PORTFOLIO

Zion GUO[1]
Hsin-Yi HUANG[2]

Abstract

A market portfolio plays an important role in many financial theories and models. It is at the heart of the capital asset pricing model and other multivariate models. Because of the market portfolio cannot be observed directly, proxy portfolios must be used to conduct empirical studies. Unfortunately, many studies found these proxies to be inefficient and even removed from the efficient frontier. According to two-fund separation theorem, we take two steps to discover the efficient market portfolio. Our thinking is straightforward and proves that our market portfolio is not only an efficient portfolio but also is situated on the capital market line. Many researches have shown that the market portfolio is extremely sensitive to performance measurements. Hence, our findings may significantly influence financial research.

Keywords: asset allocation, two-fund separation, capital market line, HJB equation, dynamic programming setting

JEL Classification: D81, G11

1. Introduction

A market portfolio is a theoretical portfolio that includes every available type of asset at a specific proportion to its market value. This concept plays an important role in many financial theories and models and is at the heart of the capital asset pricing model and other multivariate models. Because the market portfolio cannot be observed directly, proxy portfolios are used to conduct empirical studies, the most popular of which is typically a general stock market index. For example, many US studies use the CRSP equal-weighted or value-weighted index; meanwhile, most international studies use the MSCI global index. In individual countries, the FTSE100 in the UK, DAX in Germany, and the S&P500 in the US are also popular proxies for a market portfolio.

[1] *Corresponding author, Department of Business Education, College of Technology, National Changhua University of Education, 500, Changhua, Taiwan, R.O.C., Tel.: 886-4-7232105 ext.7344; Fax: 886-4-7211290, E-mail: zionguo@cc.ncue.edu.tw.*
[2] *Financial Management Center, COTA Commercial Bank, Taiwan, R.O.C.*

Many studies have examined the mean-variance efficiency of various market proxies, finding that these proxies are inefficient and typically beyond the efficient frontier.[3] Levy (1983), Green and Hollifield (1992), Huang and Satchell (2002), and Jagannathan and Ma (2003) have shown that portfolios on the efficient frontier should typically feature many short positions, implying that all of those positive-by-definition market proxies are inefficient. Besides, Roll (1978), Dybvig and Ross (1985) and Green (1986) have shown that performance measurements are extremely sensitive to market portfolio choice. Hence, identifying an efficient market portfolio is a very important matter.

Tobin (1958) first discussed the idea of two-fund separation by analyzing the portfolio demand in a mean-variance setting. Since then, much research has shown that portfolio allocation decisions may be reduced into a two-stage process. Individual investors should first decide the relative allocation across risky assets and then decide how to divide total wealth between risky assets and risk-free ones. Since significant literature already exists on portfolio separation theorems, we have chosen to refer to textbook overviews such as Ingersoll (1987) and Huang and Litzenberger (1988) to provide a detailed overview of various separation results. According to the two-fund separation theory, all investors hold a combination of risk-free assets and a market portfolio. Therefore, by way of individual investor's optimal asset allocation strategy, we may discover the market portfolio.

Over the past decades, continuous-time methods have become integral to financial economics research, important to several core areas such as consumption- portfolio selection, asset pricing theory, and options valuation. Merton (1969, 1971) was a pioneer in continuous-time modeling by formulating the intertemporal consumption and the portfolio choice problem of an individual investor. Lucas (1978) and Breeden (1979) developed a simple relationship between consumption and asset returns, contributing to the emergence of consumption-based asset pricing theory. In this paper, we first determine an individual investor's optimal asset allocation strategy in a stochastic dynamic programming setting. Then we vary the risk aversion coefficient to make the position of risk-free asset held by the individual investor equal zero. As shown in Figure 1, we first determine the optimal portfolio selection, Q, and then shift Q to M along CML (capital market line). For the time being, M is the market portfolio.

Our thinking is straightforward and proves that M is not only an efficient portfolio but also is situated on the capital market line. Since the market portfolio is extremely sensitive to performance measurements, our results are significant.

The remainder of this paper is organized as following. Section 2 expounds individual investor's utility function, section 3 depicts the economy, section 4 finds the market portfolio, section 5 proves the efficiency of the market portfolio, section 6 gives a numerical example, and section 7concludes.

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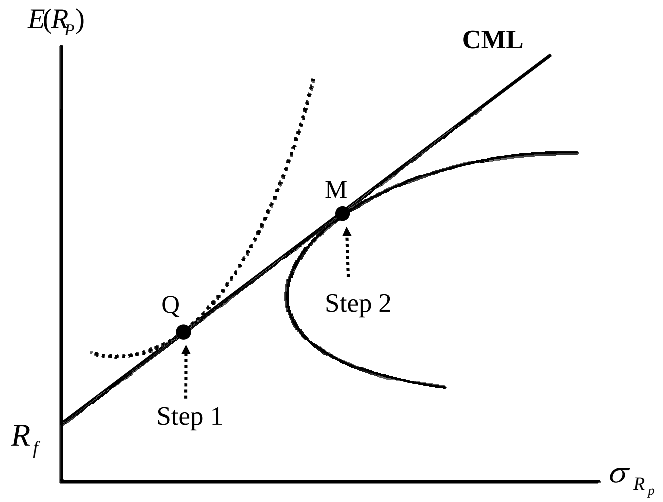
[3] *See Jobson and Korkie (1982), Shanken and Roll (1985), Kandel and Stambaugh (1987), Gibbons et al. (1989), Zhou (1991), MacKinlay and Richardson (1991), and Hwang and Satchell (2002).*

Figure 1

Two steps to determine the market portfolio



2. The utility function

To determine an individual investor's optimal asset allocation strategy in a stochastic dynamic programming setting, we must apply a utility function in advance. Cass and Stiglitz (1970) have shown that the two-fund separation holds if an investor has a HARA (hyperbolic absolute risk aversion) utility function. By two-fund separation theorem, an individual's demand for the market portfolio is proportional to his wealth. Therefore, the market portfolio should be a normal good. We ascertain whether the HARA utility function (negative exponential utility function, quadratic utility function and power utility function) regard the market portfolio as a normal good.

W_0 is assumed to be the initial wealth of an individual investor and r_M is the expected rate of return of the market portfolio. If the individual invests M dollars in the market portfolio and (W_0–M) dollars in the risk-free asset, his uncertain end of period wealth, $\tilde{W}$, would be:

$$\tilde{W} = (W_0 - M)(1 + r_f) + M(1 + r_M) = W_0(1 + r_f) + M(r_M - r_f)$$

Let U (·) be the utility function of individual investor. Then, the individual's choice problem is:

$$\underset{M}{Max}\, E\left(U\left(\tilde{W}\right)\right) = \underset{M}{Max}\, E\left(U\left(W_0(1 + r_f) + M(r_M - r_f)\right)\right) \qquad (1)$$

The first-order and second-order conditions of Equation (1) are $E\left(U'\left(\tilde{W}\right)(r_M - r_f)\right) = 0$ and $E\left(U''\left(\tilde{W}\right)(r_M - r_f)^2\right) < 0$, respectively. Hence, individual investor have maximum expected utility when $E\left(U'\left(\tilde{W}\right)(r_M - r_f)\right) = 0$. We ascertain whether the individual investor's demand of the market portfolio is proportional to wealth, i.e. $dM/dW_0 > 0$, $dM/dW_0 = 0$ or $dM/dW_0 < 0$. The implicit differentiation of *M* with respect to W_0 is:

$$\frac{dM}{dW_0} = -\frac{E\left(U''\left(\tilde{W}\right)(r_M - r_f)(1 + r_f)\right)}{E\left(U''\left(\tilde{W}\right)(r_M - r_f)^2\right)}$$

Because of $E\left(U''\left(\tilde{W}\right)(r_M - r_f)^2\right) < 0$ and $1 + r_f > 0$, we know:

$$\text{sign}\left\{\frac{dM}{dW_0}\right\} = \text{sign}\left\{E\left(U''\left(\tilde{W}\right)(r_M - r_f)\right)\right\}$$

Under decreasing *ARA* (absolute risk aversion), in the event that $r_M > r_f$, we have $\tilde{W} > W_0(1 + r_f)$ and:

$$ARA\left(\tilde{W}\right) < ARA(W_0(1 + r_f)), \quad \text{if} \quad r_M > r_f \tag{2.1}$$

In contrast, in the event that $r_M < r_f$, we have $\tilde{W} < W_0(1 + r_f)$ and:

$$ARA\left(\tilde{W}\right) > ARA(W_0(1 + r_f)), \quad \text{if} \quad r_M < r_f \tag{2.2}$$

Multiplying both sides of Equation (2.1) and (2.2) by $-U'\left(\tilde{W}\right)(r_M - r_f)$ yields:

$$U''\left(\tilde{W}\right)(r_M - r_f) > -ARA(W_0(1 + r_f))U'\left(\tilde{W}\right)(r_M - r_f), \quad \text{if} \quad r_M > r_f \tag{3.1}$$

$$U''\left(\tilde{W}\right)(r_M - r_f) > -ARA(W_0(1 + r_f))U'\left(\tilde{W}\right)(r_M - r_f), \quad \text{if} \quad r_M < r_f \tag{3.2}$$

Equation (3.1) and (3.2) imply:

$$E\left(U''\left(\tilde{W}\right)(r_M - r_f)\right) > -ARA(W_0(1 + r_f))E\left(U'\left(\tilde{W}\right)(r_M - r_f)\right) = 0$$

Under decreasing absolute risk aversion, the implicit differentiation of *M* with respect to W_0 is positive ($dM/dW_0 > 0$). Similarly, under increasing absolute risk aversion, the implicit differentiation of *M* with respect to W_0 is negative ($dM/dW_0 < 0$). Under constant absolute risk aversion, the implicit differentiation of *M* with respect to W_0 equals zero ($dM/dW_0 = 0$). Arrow (1971) shows that decreasing absolute risk aversion over the entire domain of *ARA* implies that the risky asset is a normal good, increasing absolute risk aversion implies that the risky asset is an inferior good, and constant absolute risk aversion implies that the individual's demand for the risky asset is unaffected with respect to initial wealth. We can summarize as follows:

$$\frac{dARA}{dW_0} < 0 \Rightarrow \frac{dM}{dW_0} > 0 \Rightarrow \text{The market portfolio is a normal good.}$$

$$\frac{dARA}{dW_0} > 0 \Rightarrow \frac{dM}{dW_0} < 0 \Rightarrow \text{The market portfolio is an inferior good.}$$

$$\frac{dARA}{dW_0} = 0 \Rightarrow \frac{dM}{dW_0} = 0 \Rightarrow \text{The market portfolio is a neutral good.}$$

Table 1

The characteristic of market portfolio under different utility functions

	Negative exponential utility	Quadratic utility	Power utility
$U(C)$	$-e^{-\beta C},\ \beta > 0$	$aC - \frac{b}{2}C^2$	$-C^{-\beta},\ \beta > 0$
ARA	β	$\frac{b}{a-bC}$	$\frac{1+\beta}{C}$
$\frac{dARA}{dW} = \frac{dARA}{dC}\frac{dC}{dW}$	0	$\left(\frac{b}{a-bC}\right)^2 \frac{dC}{dW} > 0$	$-\frac{1+\beta}{C^2}\frac{dC}{dW} < 0$
The characteristic of market portfolio	neutral good	inferior good	normal good

Note: $dC/dW > 0$

As shown in Table 1, only the power utility function regards the market portfolio as a normal good, but the others do not. Hence, we adopt the power utility function and take the utility function of individual investors as:

$$U(C_t) = -C_t^{-\beta} = -[C(W_t)]^{-\beta},\ \ \beta > 0 \tag{4}$$

This is a well-known strictly concave power utility function, i.e. $U'(C_t) > 0$ and $U''(C_t) < 0$.

3. The economic setting

In the market, N risky assets and one risk-free asset are assumed and all of these securities may be infinitely divided with the returns accrued only in the form of capital gains (no dividend payout). Taxes, transaction costs, and short-sell constraints are all inapplicable.

Assuming the price of the jth asset at time t, S_{jt}, follows the Ito process with the following differential equation:

$$\frac{dS_{jt}}{S_{jt}} = \mu_j dt + \sigma_j dz_j$$

where

z_j is a Wiener process.
μ_j is the expected instantaneous rate of return of the jth risky asset at time t.
σ_j is the standard deviation of expected instantaneous rate of return of the jth risky asset at time t.

Let B_t be the total amount of the risk-free asset that the investor holds at time t and the dynamics for B_t is given by:

$$\frac{dB_t}{B_t} = r_f dt$$

where r_f is the expected instantaneous rate of return of the risk-free asset.

Let W_t be the total wealth held by an investor at time t, comprising the formula:

$$W_t = \sum_{j=1}^{N} n_{jt} S_{jt} + B_t$$

where n_{jt} is the number of shares of the jth risky asset held by the investor at time t.

Then, we have the dynamic stochastic process of the total wealth:

$$dW_t = \sum_{j=1}^{N} w_{jt}\left[(\mu_j - r_f)dt + \sigma_j dz_j\right]W_t + r_f W_t dt - C_t dt \qquad (5)$$

where

C_t is the consumption of the investor at time t.
w_{jt} is the proportion of the total wealth that the investor invests in the jth risky asset at time t, j = 1,…, N .

4. The market portfolio

As shown in Figure 1, we take two steps to determine the market portfolio. In Step 1, we look for an individual investor's lifetime portfolio selection (the point Q in Figure 1) in a stochastic dynamic programming setting. In Step 2, we shift Q to M along the capital market line.

Step 1: Finding Q in Figure 1

We assume that the individual investor desires to solve the following dynamic portfolio choice problem:

$$\underset{C_t, w_t'}{Max}\, E_t\left[\int_t^T e^{-\delta\tau} U(C_\tau)\, d\tau + B(W_T, T)\right]$$

s.t. Equations (4) and (5), $C_t > 0$, $W_t > 0$.

where $B(W_T, T)$ is the bequest function.

Let $J = J(W_t, t)$ be the well-behaved function such that:

$$J = \underset{C_t, w_t'}{Max}\, E_t\left[\int_t^T e^{-\delta\tau} U(C_\tau) d\tau + B(W_T, T)\right]$$

The Hamilton-Jacobi-Bellman (HJB) equation is:

$$0 = \underset{C_t, \mathrm{w}_t'}{Max}\left\{e^{-\delta t}U(C_t) + J_W(\mathrm{w}_t'\mu_t W_t + r_f W_t - C_t) + J_t + \frac{1}{2}J_{WW}\mathrm{w}_t'\Sigma_t \mathrm{w}_t W_t^2\right\} \tag{6}$$

where

w_t is the $N \times 1$ vector with representative elements w_{jt}.

w_t' is the transpose of w_t.

μ_t is the $N \times 1$ vector of expected instantaneous rate of excess return of risky assets at time *t.*

Σ_t is the $N \times N$ variance-covariance matrix of the expected instantaneous rate of return of risky assets at time *t*.

J_W denotes the derivative of *J* with respect to W_t , with a similar notation used for higher derivatives.

J_t denotes the derivative of *J* with respect to *t*.

The first order conditions to Equation (6) are:

$$\begin{cases} U_{C_t} = J_W \\ \mathrm{w}_t = -\dfrac{J_W}{J_{WW}W_t}\Sigma_t^{-1}\mu_t \end{cases}$$

As showed in Merton (1969), we have the following optimal asset allocation strategy for the investor:

$$\mathrm{w}_t = \frac{1}{1+\beta}\Sigma_t^{-1}\mu_t$$

Step 2: Shift Q to M along CML

After determining the investor’s optimal asset allocation strategy, we alter the risk aversion coefficient to make $\mathrm{w}_t'1 = 1$ (**1** is the $N \times 1$ vector with representative elements 1). This implies that $1+\beta = \mu_t'\Sigma_t^{-1}1$ and the investor holds a zero share of risk-free assets. At this moment, the investor invests all wealth into a market portfolio, which is shown as:

$$\mathrm{w}_M = \frac{\Sigma_t^{-1}\mu_t}{\mu_t'\Sigma_t^{-1}1} \tag{7}$$

Our thinking is straightforward and the market portfolio determined in our model is definitely not a positive-by-definition market portfolio. In the next section, we confirm the efficiency of the market portfolio.

5. Proving the efficiency of our market portfolio

The market portfolio is a unique efficient frontier portfolio situated on the capital market line. Thus, we must prove two matters: 1) that our market portfolio is an efficient frontier portfolio and 2) that our market portfolio is situated on the capital market line.

Our market portfolio is an efficient frontier portfolio

A portfolio is a frontier portfolio if it has the minimum variance among portfolios with the same expected rate of return. In the mean-variance efficiency world, the investor selects the following weighting vector to minimize risk (variance) for given expected return $E_t(r_p)$.

$$\begin{aligned} &\underset{\mathrm{w}}{Min} \ \frac{1}{2}\mathrm{w}'\Sigma_t^{-1}\mathrm{w} \\ s.t. \ &\mathrm{w}'(\mu_t + r_f 1) = E_t(r_p) \\ &\mathrm{w}'1 = 1 \end{aligned}$$

Forming the Lagrangian, and let $\mathbf{w_p}$ be the solution to the following:

$$\underset{\mathrm{w},\lambda,\gamma}{Min} \ \mathrm{L} = \frac{1}{2}\mathrm{w}'\Sigma_t^{-1}\mathrm{w} + \lambda\left(E_t(r_p) - \mathrm{w}'(\mu_t + r_f 1)\right) + \gamma(1 - \mathrm{w}'1)$$

where λ and γ are two positive constants.

The unique set of portfolio weights for the frontier portfolio having an expected rate of return $E_t(r_p)$ is:

$$\mathrm{w_p} = g + hE_t(r_p) \tag{8}$$

where

$$g = \frac{1}{D}\left(B\Sigma_t^{-1}1 - A\Sigma_t^{-1}(\mu_t + r_f 1)\right)$$

$$h = \frac{1}{D}\left(C\Sigma_t^{-1}(\mu_t + r_f 1) - A\Sigma_t^{-1}1\right)$$

$$A = (\mu_t + r_f 1)'\Sigma_t^{-1}1 = \mu_t'\Sigma_t^{-1}1 + r_f 1'\Sigma_t^{-1}1 = a + r_f C$$

$$B = (\mu_t + r_f 1)'\Sigma_t^{-1}(\mu_t + r_f 1) = \mu_t'\Sigma_t^{-1}\mu_t + r_f\mu_t'\Sigma_t^{-1}1 + r_f 1'\Sigma_t^{-1}\mu_t + r_f^2 1'\Sigma_t^{-1}1 = b + 2r_f a + r_f^2 C$$

$$C = 1'\Sigma_t^{-1}1$$

$$D = BC - A^2 = bc - a^2$$

$$a = \mu_t'\Sigma_t^{-1}1 = 1'\Sigma_t^{-1}\mu_t$$

$$b = \mu_t'\Sigma_t^{-1}\mu_t$$

Concerning our model, the expected rate of return of our market portfolio is represented as:

$$E_t(r_M) = \mathrm{w_t}'(\mu_t + r_f 1) = \frac{b}{a} + r_f \tag{9}$$

Given the expected rate of return, $E_t(r_M) = \frac{b}{a} + r_f$, the unique frontier portfolio is:

$$\begin{aligned} \mathrm{w_p} &= g + hE_t(r_M) \\ &= \frac{1}{D}\left(B\Sigma_t^{-1}1 - A\Sigma_t^{-1}(\mu_t + r_f 1)\right) + \frac{1}{D}\left(C\Sigma_t^{-1}(\mu_t + r_f 1) - A\Sigma_t^{-1}1\right)\left(\frac{b}{a} + r_f\right) \\ &= \frac{1}{D}\left(\left(b + 2r_f a + r_f^2 C\right)\Sigma_t^{-1}1 - (a + r_f C)\Sigma_t^{-1}\mu_t - \left(r_f a + r_f^2 C\right)\Sigma_t^{-1}1\right) \end{aligned}$$

$$+\frac{b}{aD}\left(C\Sigma_t^{-1}\mu_t + r_f C\Sigma_t^{-1}1 - (a + r_f C)\Sigma_t^{-1}1\right) + \frac{r}{D}\left(C\Sigma_t^{-1}\mu_t + r_f C\Sigma_t^{-1}1 - (a + r_f C)\Sigma_t^{-1}1\right)$$
$$= \frac{1}{D}\left(\frac{bC}{a} - a\right)\Sigma_t^{-1}\mu_t = \frac{\Sigma_t^{-1}\mu_t}{a} = \frac{\Sigma_t^{-1}\mu_t}{\mu_t'\Sigma_t^{-1}1}$$
$$= w_M$$

Hence, our market portfolio shown in Equation (7) is a frontier portfolio. Since a frontier portfolio is not necessarily an efficient one, we must further prove that our market portfolio is also efficient. Huang and Litzenberger (1988) demonstrate that the set of all frontier portfolios is called the portfolio frontier, one important property of which is that for any portfolio p on the frontier, except for the minimum variance portfolio, there exists an unique frontier portfolio, denoted by $ZC(p)$, featuring a zero covariance with p. If p is an efficient portfolio, then $ZC(p)$ is an inefficient portfolio, and vice versa. Therefore, frontier portfolios, which have expected rates of return strictly higher than that of the minimum variance portfolio, are efficient portfolios.

Let p be an arbitrary frontier portfolio. By Equation (8), the variance of the rate of return of portfolio p is:

$$\sigma^2(r_p) = w_p'\Sigma_t w_p = \frac{C}{D}\left(E_t(r_p) - \frac{A}{C}\right)^2 + \frac{1}{C}$$

The first-order and second-order differential of $\sigma(r_p)$ with respect to $E_t(r_p)$ are $d\sigma(r_p)/dE_t(r_p) = C\,(E_t(r_p) - A/C)/D\sigma(r_p)$ and $d^2\sigma(r_p)/dE_t(r_p)^2 = 1/D\sigma(r_p)^3 > 0$, respectively. Apparently, p is the minimum variance portfolio when $E_t(r_p) = A/C$. By Equation (8), the covariance between our market portfolio M and its corresponding zero covariance portfolio $ZC(M)$ is:

$$Cov(r_M, r_{ZC(M)}) = w_M'\Sigma_t w_{ZC(M)} = \frac{C}{D}\left(E_t(r_M) - \frac{A}{C}\right)\left(E_t(r_{ZC(M)}) - \frac{A}{C}\right) + \frac{1}{C} = 0$$

Replacing Equation (9) into above equation, we have the expected rate of return of $ZC(M)$ as:

$$E_t(r_{ZC(M)}) = \frac{A}{C} - \frac{\dfrac{D}{C^2}}{E_t(r_M) - \dfrac{A}{C}} = r_f$$

Because of r_f is less than the expected rate of return of minimum variance portfolio, $\frac{A}{C}$, $ZC(M)$ is an inefficient portfolio. Therefore, our market portfolio is an efficient portfolio.

Our market portfolio is situated on the capital market line

The variance of our market portfolio is:

$$\sigma^2(r_M) = w_M'\Sigma_t w_M = \frac{C}{D}\left(E_t(r_M) - \frac{A}{C}\right)^2 + \frac{1}{C} = \frac{b}{a^2}$$

The tangent to our market portfolio is:

$$\frac{dE_t(r_M)}{d\sigma(r_M)} = \frac{D\sigma(r_M)}{CE_t(r_M) - A}$$

The intercept of above tangent is:

$$E_t(r_M) - \frac{dE_t(r_M)}{d\sigma(r_M)}\sigma(r_M) = E_t(r_M) - \frac{D\sigma(r_M)^2}{CE_t(r_M) - A} = r_f$$

Therefore, our market portfolio is clearly situated on the capital market line. Since the market portfolio is extremely sensitive to performance measurements, our findings may significantly influence financial research.

6. A numerical example

Assume one risk-free asset and two risky assets in the economy. The instantaneous rate of daily return of the risk-free asset is $r_f = 0.0002$, and the mean vector and covariance matrix of the expected instantaneous rate of excess daily return of the two risky assets are:

$$\mu_t = \begin{bmatrix} 0.0002 \\ 0.0008 \end{bmatrix}, \quad \Sigma_t = \begin{bmatrix} 0.00016 & 0.00008 \\ 0.00008 & 0.00060 \end{bmatrix}$$

By Equation (7), we have the market portfolio $w_M = \Sigma_t^{-1}\mu_t / \mu_t'\Sigma_t^{-1}1$. It means that the coefficient of relative risk aversion $1+\beta = \mu_t'\Sigma_t^{-1}1 = 1.875$.

Table 2

Relative risk aversion coefficient vs. optimal asset allocation strategy

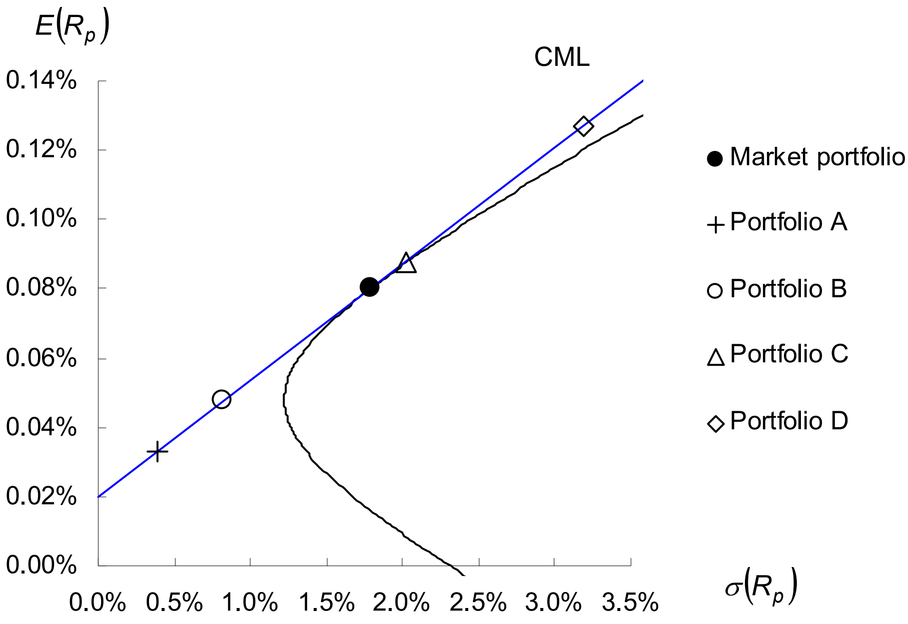
	Market Portfolio	Portfolio A	Portfolio B	Portfolio C	Portfolio D
$1+\beta$	1.875	8.692	4.084	1.660	1.054
Risk-free asset	0.000	0.784	0.541	-0.130	-0.779
Risky 1	0.333	0.072	0.153	0.377	0.593
Risky 2	0.667	0.144	0.306	0.753	1.186
Expected return	0.08%	0.03%	0.05%	0.09%	0.13%
Std. dev.	1.79%	0.39%	0.82%	2.02%	3.18%

Table 2 shows, in our numerical example, the market portfolio involves 33.3% of the first risky asset and 66.7% of the second risky asset. The mean return and standard deviation of the market portfolio is 0.08% and 1.79%, respectively. If the relative risk aversion coefficient of the investor is 8.692, as shown in Table 2, portfolio A is the best choice for the investor. If the relative risk aversion coefficient of the investor is 4.084, portfolio B is suitable for the investor. Table 2 also shows that as the coefficient of risk aversion ($1+\beta$) declines, less is invested in the risk-free asset and more in the two risky assets. In portfolio C and D, the investor leverages the portfolio by taking a short position in the risk-free asset and increasing long positions in the two risky assets.

Figure 2 depicts the portfolio frontier, the capital market line, and the portfolios show in Table 2. Portfolios A, B, C, and D are all situated on the capital market line. Therefore, investors can make sure that all of the entire candidate portfolios are efficient portfolios. It is consistent with the two-fund separation theory that all investors hold a combination of risk-free assets and a market portfolio.

Figure 2

Optimal and efficient portfolios



7. Conclusion

The market portfolio is a theoretical portfolio that includes every available type of asset at a specific proportion to its market value. Because of the market portfolio cannot be observed directly, it is necessary to use proxy portfolios when conducting empirical studies. The most popular proxy for the market portfolio is typically a general stock market index. For example, the CRSP equal-weighted or value-weighted index and the S&P500 in the US, the FTSE100 in the UK, the DAX in Germany, and the MSCI global index are popular proxies for a market portfolio. Since portfolios on the efficient frontier should typically feature many short positions, implying that all of those positive-by-definition market proxies are inefficient and even typically far from the efficient frontier. By two-fund separation theorem, an individual's demand for the market portfolio is proportional to his wealth. After determining the investor's optimal asset allocation strategy, we alter the risk aversion coefficient to make the investor holds a zero share of risk-free assets. At this moment, the investor invests all wealth into a market portfolio. Our thinking is straightforward and the market portfolio determined in our model is definitely not a positive-by-definition market portfolio. We prove that our market portfolio is not only efficient but also is situated on the capital market line. Since the market portfolio is extremely sensitive to performance measurements, our findings may significantly influence financial research.

References

Arrow, K. J., 1971. Essays in the Theory of Risk Bearing. Amsterdam: North-Holland Publishing Co.

Breeden, D. T., 1979. An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities. *Journal of Financial Economics*, 7(3), pp. 265-296.

Cass, D. and J. E. Stiglitz, 1970. The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds. *Journal of Economic Theory*, 2, pp. 122-160.

Dybvig, P. H. and S. A. Ross, 1985. The Analytics of Performance Measurement Using a Security Market Line. *Journal of Finance*, 40(2), pp. 401-416.

Gibbons, M. R., S. A. Ross and J. Shanken, 1989. A Test of the Efficiency of a Given Portfolio. *Econometrica*, 57(5), pp. 1121-1152.

Green, R. C., 1986. Benchmark Portfolio Inefficiency and Deviations from the Security Market Line. *Journal of Finance*, 41(2), pp. 295-312.

Green, R. C. and B. Hollifield, 1992. When Will Mean-Variance Efficient Portfolios be Well Diversified? *Journal of Finance*, 47(5), pp. 1785-1809.

Huang, Chi-fu and R. H. Litzenberger, 1988. Foundations of Financial Economics. North-Holland, New York, NY.

Hwang, S. and S. E. Satchell, 2002. Calculating the Misspecification in Beta from Using a Proxy for the Market Portfolio. *Applied Financial Economics*, 12(11), pp. 771-781.

Ingersoll, J. E. Jr., 1987. Theory of Financial Decision Making. Rowman & Littlefield, Savage, Maryland.

Jagannathan, R. and T. Ma, 2003. Risk-Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps. *Journal of Finance*, 58(4), pp. 1651-1684.

Jobson, J. D. and B. Korkie, 1982. Potential Performance and Tests of Portfolio Efficiency. Journal *of Financial Economics*, 10(4), pp. 433-466.

Kandel, S. and R. F. Stambaugh, 1987. On Correlations and Inferences about Mean-Variance Efficiency. *Journal of Financial Economics*, 18(1), pp. 61-90.

Levy, H., 1983. The Capital Asset Pricing Model: Theory and Empiricism. *Economic Journal*, 93(369), pp. 145-165.

Lucas, R. E. Jr., 1978. Asset Prices in an Exchange Economy. *Econometrica*, 46(6), pp. 1429-1445.

MacKinlay A. C. and M. P. Richardson, 1991. Using Generalized Method of Moments to Test Mean-Variance Efficiency. *Journal of Finance*, 46(2), pp.-511-527.

Merton, R. C., 1969. Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case. *Review of Economics and Statistics*, 51(3), pp. 247-257.

Merton, R. C., 1971. Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory*, 3(4), pp. 373-413.

Roll R., 1978. Ambiguity when Performance is Measured by the Security Market Line. *Journal of Finance*, 33(4), pp. 1051-1069.

Shanken, J. and R. Roll, 1985. Multivariate Tests of the Zero-Beta CAPM/A Note on the Geometry of Shanken's CSR T (superscript 2) Test for Mean/Variance Efficiency. *Journal of Financial Economics*, 14(3), pp. 327-357.

Tobin, J., 1958. Liquidity Preferences as Behavior Towards Risk. *Review of Economic Studies*, 25(2), pp. 65-86.

Zhou, G., 1991. Small Sample Tests of Portfolio Efficiency. *Journal of Financial Economics*, 30(1), pp. 165-191.