

6. FORECASTING VALUE-AT-RISK WITH TWO-STEP METHOD: GARCH-EXPONENTIATED ODD LOG-LOGISTIC NORMAL MODEL

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Abstract

The purpose of this study is to evaluate the forecasting ability of GARCH-type models in estimating the Value-at-Risk (VaR) by introducing a new four-parameter distribution, called Exponentiated Odd Log-Logistic Normal distribution. The statistical properties of new heavy-tailed distribution are investigated and a simulation study is performed to assess the maximum likelihood estimations of introduced distribution. Then, the VaR is forecasted by using mean and volatility forecasts and quantile estimation of introduced distribution. Daily VaR forecasting ability of proposed two-stage model is compared with the GARCH models specified under heavy-tailed distributions by means of two backtesting methods. Empirical findings show that proposed two-stage model outperforms to well-known distributions such as normal, Student's-t, generalized error, and skewed generalized error distributions at high quantiles.

Keywords: Value-at-Risk, GARCH model, log-logistic distribution, maximum likelihood, estimation, normal distribution

JEL Classification: C46, C53, G32

1. Introduction

The Value at Risk (VaR) is one of the most popular approaches to measure market risk. The VaR, in its most general form, measures the potential loss in value of a risky asset or portfolio over a defined period for a given confidence interval. From a statistical point of view, the VaR entails the estimation of quantile of the distribution of returns. Despite its importance and simplicity, there is no universally accepted method to compute the VaR of a portfolio,

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while different models may lead to significantly different risk measures (Kuester et al., 2006; McMillian and Kambouroudis, 2009). A main concern in the estimation of market risk with the VaR method is the choice of the appropriate model, since a misspecified model leads to inaccurate risk estimation.

While calculating the VaR using one of the statistical models, many assumptions are necessary. One of them is that the asset returns are identically, independent and normal distributed. However, in reality, the financial data is not normal distributed and exhibits properties of skewness or kurtosis. Therefore, modeling VaR with normality assumption, without considering the big and unexpected losses that stated in tail of the distribution, causes underestimated or overestimated VaR forecasts. Because of the fact that normal distribution fails to modeling the tail of the financial return series, many researchers have used skewed and heavy-tailed distributions in forecasting VaR to overcome this problem. Numerous models have been suggested to capture the volatility clustering effect; the most widely used one is the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model (Bollerslev, 1986). Nelson (1991), Zangari (1996), Venkataraman (1997), Angelidis et al. (2004), Harmantzis et al. (2004), Hung et al. (2008), Lee et al. (2008), and Braione and Scholtes (2016) evaluated the performance of GARCH models under heavy-tailed distributions, such as Student-t, mixture normal, generalized error distribution, skewed generalized error distribution, to forecast daily VaR. As a result of these studies, leptokurtic distributions are able to produce better daily VaR forecasts due to financial return series exhibit skewness and excess kurtosis. However, more flexible distribution is still required for estimating VaR as a statistical tool for advance risk modelling.

More concretely, our objective is to examine the suitability of GARCH models by proposing exponentiated odd log-logistic normal (EOLLN) distribution which yields quite satisfactory results. For this aim, we first introduce a four-parameter distribution, called as EOLLN, which extends the normal distribution and provides skewed and heavy-tailed structures for shape of the probability density function (pdf). The main advantage of the proposed distribution for modeling VaR is to provide a better fitting performance to financial return series than other well-known distributions such as normal, Student's-t, generalized error, and skewed generalized error distributions. This distribution is a good candidate to remove the inability of well-known distributions modeling the tail of the financial return distribution. Then, we define a dynamic VaR model, named GARCH-EOLLN, to forecast the daily VaR based on the proposed distribution with GARCH volatility model. The proposed two step VaR model can be summarized as follows: In the first step, financial return series is modeled by GARCH volatility model to obtain one-day-ahead forecasts of conditional mean, volatility and extract the standardized residuals. In the second step, extracted standardized residuals are modeled by introduced distribution to obtain quantile estimation for given confidence levels.

The rest of the paper is organized as follows: Section 2 introduces the statistical properties of EOLLN distribution comprehensively. GARCH models based on different distributional assumptions and proposed model specification are presented in Section 3. Backtesting methodology is given in Section 4. Empirical findings and model comparisons are presented in Section 5. Conclusion is given in Section 6.

II. Exponentiated Odd Log-logistic Normal Distribution

In this section, we propose a new extended normal distribution with heavier tails called the exponentiated odd log-logistic normal (EOLLN) model.

Braga et al. (2016) recently proposed odd log-logistic normal (OLLN) distribution with shape parameter $\alpha > 0$ which is much more flexible than some well-known distributions such as beta normal, skew normal, gamma normal, and Kumaraswamy normal models. The cdf of the EOLLN model with an additional shape parameter $\beta > 0$ is defined by

$$F(x; \alpha, \beta, \mu, \sigma) = \left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)^\alpha}{\Phi\left(\frac{x-\mu}{\sigma}\right)^\alpha + \left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right]^\alpha} \right]^\beta \tag{1}$$

where $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter, $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively.

The corresponding EOLLN density is given by

$$f(x; \alpha, \beta, \mu, \sigma) = \frac{\alpha \beta \phi\left(\frac{x-\mu}{\sigma}\right) \Phi\left(\frac{x-\mu}{\sigma}\right)^{\alpha\beta-1} \left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\alpha-1}}{\sigma \left\{ \Phi\left(\frac{x-\mu}{\sigma}\right)^\alpha + \left[1 - \Phi\left(\frac{x-\mu}{\sigma}\right)\right]^\alpha \right\}^{\beta+1}} \tag{2}$$

where $\alpha > 0$ and $\beta > 0$ are the shape parameters. Henceforth, a random variable with density function (2) is denoted by $X \sim \text{EOLLN}(\alpha, \beta, \mu, \sigma)$.

Plots of the EOLLN density function for some parameter values are displayed in Figure 1. Based on Figure 1, we compare the tail behavior of EOLLN distribution with normal distribution. It is clear that EOLLN distribution has fatter tail than normal distribution for different parameter combinations. This property of EOLLN distribution provides an opportunity to estimate more realistic tail probability than normal distribution for financial return series.

Equation (3) has tractable properties especially for simulations. The tail probabilities of EOLLN distribution are compared with normal distribution. Table 1 shows the tail probabilities of EOLLN and normal distributions for some selected parameter values. Based on Table 1, it is clear that EOLLN distribution has fatter tails than normal distribution.

The quantile function (qf) is in widespread use in statistics. Let $F(x; \alpha, \beta, \mu, \sigma) = u$ and $\Phi^{-1}\left(\frac{x-\mu}{\sigma}\right)$ be the inverse of $\Phi\left(\frac{x-\mu}{\sigma}\right)$. Then, the qf of X is given by

$$Q(u) = \mu + \sigma \Phi^{-1} \left[\frac{u^{\frac{1}{\alpha\beta}}}{u^{\frac{1}{\alpha\beta}} + \left(1 - u^{\frac{1}{\alpha\beta}}\right)^{\frac{1}{\alpha}}} \right] \tag{3}$$

Figure 1

Plot of the EOLLN pdf for Several Values of Parameters

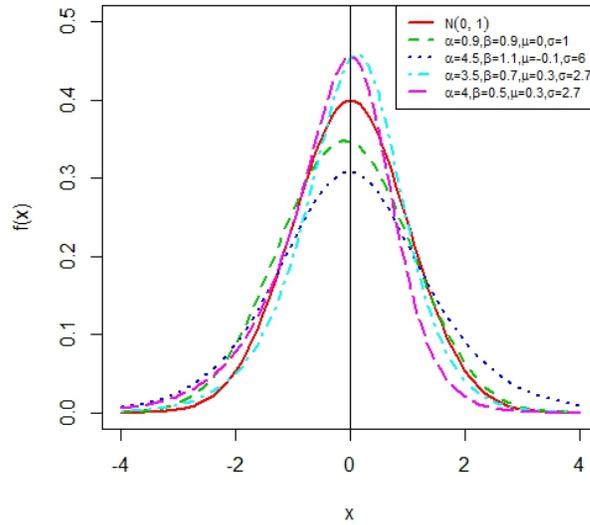


Table 1

Left and Right Tail Probabilities of Normal and EOLLN Distributions

Distribution	P(X>2)	P(X>3)	Distribution	P(X<-2)	P(X<-3)
Normal(0,1)	0.0227	0.0013	Normal(0,1)	0.0227	0.0013
EOLLN(0.5,1.5,0,1)	0.1918	0.0527	EOLLN(0.5,1.5,0,1)	0.0481	0.0067
Normal(2,2)	0.5000	0.3085	Normal(0,2)	0.1586	0.0668
EOLLN(1.5,2.5,2,2)	0.8232	0.479	EOLLN(1.5,0.5,0,2)	0.2751	0.1371
Normal(-1,5)	0.2742	0.2118	Normal(0,4)	0.3085	0.2266
EOLLN(0.7,2,-1,5)	0.5590	0.4888	EOLLN(0.5,0.8,0,4)	0.4809	0.4329
Normal(-2,3)	0.0912	0.0477	Normal(0,5)	0.3445	0.2742
EOLLN(0.5,0.7,-2,3)	0.1752	0.1319	EOLLN(0.9,0.7,0,5)	0.488	0.4245

Estimation

In this subsection, we consider estimation of the unknown parameters of the EOLLN distribution by the method of maximum likelihood. Let x_1, x_2, \dots, x_n be observed values from the EOLLN distribution with parameters $(\alpha, \beta, \mu, \sigma)$. The log-likelihood function for $(\alpha, \beta, \mu, \sigma)$ is given by

$$\ell = n \log \left(\frac{\alpha}{\sigma \sqrt{2\pi}} \right) - \frac{1}{2} \sum_{i=1}^n \left(\frac{x_i - \mu}{\sigma} \right)^2 + (\alpha\beta - 1) \sum_{i=1}^n \log(t_i) + (\alpha - 1) \sum_{i=1}^n \log(1 - t_i) - (\beta + 1) \sum_{i=1}^n \log [t_i^\alpha + (1 - t_i)^\alpha]$$

where $t_i = \Phi \left(\frac{x_i - \mu}{\sigma} \right)$.

The derivatives of the log-likelihood function with respect to the parameters $(\alpha, \beta, \mu, \sigma)$ are given by, respectively,

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \beta \sum_{i=1}^n \log(t_i) + \sum_{i=1}^n \log(1-t_i) - (\beta+1) \sum_{i=1}^n \frac{t_i^\alpha \log(t_i) + (1-t_i)^\alpha \log(1-t_i)}{t_i^\alpha + (1-t_i)^\alpha} \\ \frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \alpha \sum_{i=1}^n \log(t_i) - \sum_{i=1}^n \log[t_i^\alpha + (1-t_i)^\alpha] \\ \frac{\partial \ell}{\partial \mu} &= -\sum_{i=1}^n \frac{(x_i - \mu)}{\sigma^2} + (\alpha\beta - 1) \sum_{i=1}^n \frac{t_i^{(\mu)}}{t_i} + (1-\alpha) \sum_{i=1}^n \frac{t_i^{(\mu)}}{1-t_i} - \alpha(\beta+1) \sum_{i=1}^n \frac{t_i^{(\mu)} t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^\alpha + (1-t_i)^\alpha} \\ \frac{\partial \ell}{\partial \sigma} &= \frac{-n}{\sigma} + \sum_{i=1}^n \frac{(x_i - \mu)^2}{\sigma^3} + (\alpha\beta - 1) \sum_{i=1}^n \frac{t_i^{(\sigma)}}{t_i} + (1-\alpha) \sum_{i=1}^n \frac{t_i^{(\sigma)}}{1-t_i} - \alpha(\beta+1) \sum_{i=1}^n \frac{t_i^{(\sigma)} t_i^{\alpha-1} - (1-t_i)^{\alpha-1}}{t_i^\alpha + (1-t_i)^\alpha} \end{aligned}$$

Where $t_i^{(\mu)} = \frac{-1}{\sigma} \Phi\left(\frac{x_i - \mu}{\sigma}\right)$ and $t_i^{(\sigma)} = \frac{-(x_i - \mu)}{\sigma^2} \Phi\left(\frac{x_i - \mu}{\sigma}\right)$. The MLEs of $(\alpha, \beta, \mu, \sigma)$, say

$(\hat{\alpha}, \hat{\beta}, \hat{\mu}, \hat{\sigma})$, are the simultaneous solutions of the equations $\frac{\partial \log L}{\partial \alpha} = 0$, $\frac{\partial \log L}{\partial \beta} = 0$,

$\frac{\partial \log L}{\partial \sigma} = 0$, and $\frac{\partial \log L}{\partial \mu} = 0$.

Simulation Study

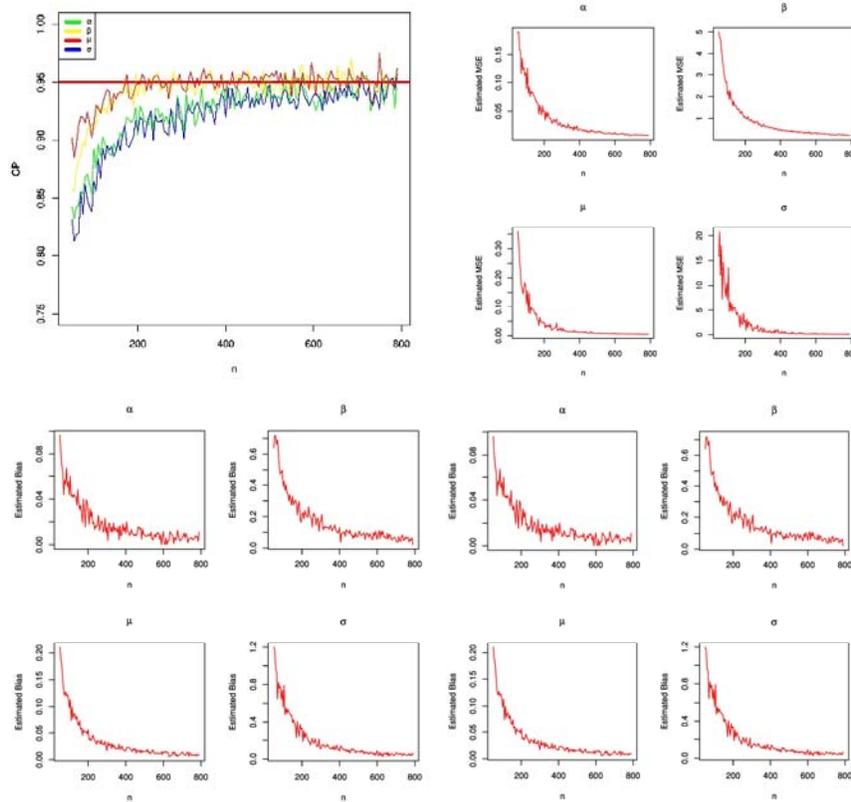
Now, we evaluate the performance of the MLEs of the parameters of EOLLN model with a simulation study. Inverse transform algorithm is used to generate random data from the EOLLN distribution. The precision of the MLEs is discussed by means of bias, mean square error (MSE), estimated average length (AL) and coverage probability (CP). We generated $N=1000$ samples of sizes $n = 50, 55, \dots, 800$ from EOLLN distribution with $\alpha = 3.5, \beta = 0.7, \mu = 0.3, \sigma = 2.7$. We obtained MLEs of the parameters for each generated sample and standard errors of MLEs are obtained by inverting observed information matrix. The estimated bias, MSEs CPs and ALs are obtained using following equations:

$$\begin{aligned} \overline{Bias}_\alpha(n) &= \frac{1}{N} \sum_{i=1}^N (\hat{\alpha}_i - \alpha) \\ \overline{MSE}_\alpha(n) &= \frac{1}{N} \sum_{i=1}^N (\hat{\alpha}_i - \alpha)^2 \\ CP_\alpha(n) &= \sum_{i=1}^N I(\hat{\alpha}_i - 1.95996s_{\hat{\alpha}_i}, \hat{\alpha}_i + 1.95996s_{\hat{\alpha}_i}) \\ AL_\alpha(n) &= \frac{3.919928}{N} \sum_{i=1}^N s_{\hat{\alpha}_i} \end{aligned}$$

where $\alpha = (\alpha, \beta, \mu, \sigma)$. The numerical results of simulation are shown in the plots of Figures 2. It is clear from these plots that the estimated biases and MSEs decrease when the sample n increases. The CPs of all parameters are near to 0.95 and approaches to nominal value when the sample size increases. Further, the AL of all parameters decreases when the

parameters decreases when the sample size increases. The results are obtained for selected parameters but similar results can be obtained for other parameter combinations.

Figure 2
Estimated Biases, MSEs, ALs and CPs for Selected Parameter Values



III. GARCH Models in VaR Estimation and Proposed Model Specification

The VaR of a long position (left tail of the distribution function) over a given time horizon t and probability p , while p is one minus the VaR confidence level, is defined as

$$VaR_p = F^{-1}(1-p)$$

where F is the distribution function of financial losses, F^{-1} denotes the inverse of F and P is the quantile at which VaR is calculated. ARCH(q) by and GARCH(p,q) models are used to model time-varying volatility in financial econometrics. ARCH(q) model was introduced by

Engle (1982) and expressed the conditional volatility as a linear function of the past q squared residuals. GARCH(p, q) model was introduced by Bollerslev (1986) and expressed the conditional volatility as a linear function of the past q squared residuals and past p conditional volatilities. In order to estimate VaR, let $R_t = \ln(S_t/S_{t-1}) \times 100$ denotes the daily returns of the assets on time t and S_t represents represents the closed prices of the assets. GARCH(1,1) can be written as follows:

$$\begin{aligned} R_t &= \mu_t + e_t \\ e_t &= \epsilon_t \sigma_t \quad \epsilon_t \sim iid. \\ \sigma_t^2 &= \omega + \alpha e_{t-1}^2 + \beta \sigma_{t-1}^2 \end{aligned}$$

Where: μ_t and σ_t^2 respectively, are the conditional mean and variance. ϵ_t is the innovation distribution and commonly supposed that the innovation distribution follows a normal distribution. Log-likelihood function of GARCH model under normality assumption can be written as follows:

$$L(\psi) = -0.5 \left(T \ln 2\pi + \sum_{t=1}^T \ln \sigma_t^2 + \sum_{t=1}^T \epsilon_t^2 \right)$$

According to GARCH-N model, one-day-ahead VaR forecast can be calculated as:

$$VaR_{t+1} = \mu_{t+1} + \phi_p^{-1}(\epsilon_t) \sigma_{t+1}$$

Where: $\phi_p^{-1}(\epsilon_t)$ is the quantile of standard normal distribution at p level and respectively μ_{t+1} , σ_{t+1} are one-day-ahead forecasts of the conditional mean and variance.

GARCH-models coupled with conditionally normally distributed innovations “conditional student distribution” is unable to fully account for the tails of the marginal distributions of daily returns. Several conditional distributions proposed in the GARCH (e.g., Student’s-t distribution, skewed generalized error distribution (SGED) and generalized error distribution (GED)). Bollerslev (1986, 1987) proposed the standardized Student’s-t distribution with $\nu > 2$ degree of freedom. Student’s-t is symmetric distribution and for $\nu > 4$, conditional kurtosis greater than 3, which exceeds the normal value. Under this specification, the log-likelihood function, for a sample of T observations, is given by

$$L(\psi) = T \left[\ln \Gamma \left(\frac{\nu+1}{2} \right) - \ln \Gamma \left(\frac{\nu}{2} \right) - \frac{1}{2} \ln [\pi(\nu-2)] \right] - \frac{1}{2} \sum_{t=1}^T \left[\ln \sigma_t^2 + (1+\nu) \ln \left(1 + \frac{\epsilon_t^2}{\nu-2} \right) \right]$$

Where: $\Gamma(\nu)$ is the gamma function and ν is the thickness parameter of the distribution tails. The one-day-ahead VaR forecast based on Student’s-t distribution is defined as

$$VaR_{t+1} = \mu_{t+1} + \phi_p^{-1}(\epsilon_t) \sigma_{t+1}$$

where $\phi_p^{-1}(\epsilon_t)$ is the quantile of Student’s-t distribution at p level and respectively μ_{t+1} , σ_{t+1} are one-day-ahead forecasts of the conditional mean and variance.

In order to model the excess kurtosis observed asset prices, assumption on ε_t can be relaxed. Nelson (1991) proposed the GED instead of assuming ε_t is normally distributed. Under this specification, the log-likelihood function for GED distributed ε_t :

$$L(\psi) = \sum_{t=1}^T \left[\ln\left(\frac{\nu}{2}\right) - \frac{1}{2} \left| \frac{\varepsilon_t}{\lambda} \right|^\nu - (1+\nu^{-1}) \ln(2) - \ln \Gamma\left(\frac{1}{2}\right) - \frac{1}{2} \ln(\sigma_t^2) \right]$$

where: ν is the tail-thickness parameter and

$$\lambda = \left(\frac{\Gamma\left(\frac{1}{\nu}\right)}{2^{\frac{2}{\nu}} \Gamma\left(\frac{3}{\nu}\right)} \right)^{\frac{1}{2}}$$

Here, $\Gamma(\cdot)$ is the gamma function. The Gaussian distribution is a special case of the GED when $\nu = 2$. If $\nu < 2$, the GED has fatter tails than Gaussian distribution. According to Nelson (1991) specification, the log-likelihood function is given by

$$L(\psi) = \sum_{t=1}^T \left[\ln\left(\frac{\nu}{\lambda}\right) - \frac{1}{2} \left| \frac{\varepsilon_t}{\lambda} \right|^\nu - (1+\nu^{-1}) \ln(2) - \ln \Gamma\left(\frac{1}{\nu}\right) - \frac{1}{2} \ln(\sigma_t^2) \right]$$

According to GARCH(1,1) model, $\sigma_t^2 = \omega + \alpha e_{t-1}^2 + \beta \sigma_{t-1}^2$ in the above equation. Parameters of the GARCH(1,1) model can be obtained by the numerical maximization procedure. The one-day-ahead VaR forecast based on GED is defined as

$$VaR_{t+1} = \mu_{t+1} + \phi_p^{-1}(\varepsilon_t) \sigma_{t+1}$$

where: $\phi_p^{-1}(\varepsilon_t)$ is the left quantile of the GED at p level.

Lee et al. (2008) used the SGED which provides a flexible distribution for modeling the empirical distribution of financial data. The pdf of standardized SGED is given by

$$f(\varepsilon_t) = C \exp\left(- \frac{|\varepsilon_t + \delta|^\kappa}{[1 + \text{sign}(\varepsilon_t + \delta)\lambda]^\kappa \theta^\kappa} \right)$$

where:

$$C = \frac{\kappa}{2\theta} \Gamma\left(\frac{1}{\kappa}\right)^{-1}, \theta = \Gamma\left(\frac{1}{\kappa}\right)^{0.5} \Gamma\left(\frac{3}{\kappa}\right)^{0.5} S(\lambda)^{-1},$$

$$S(\lambda) = \sqrt{1 + 3\lambda^2 - 4A^2\lambda^2}, \delta = \frac{2\lambda A}{S(\lambda)},$$

$$A = \Gamma\left(\frac{2}{\kappa}\right) \Gamma\left(\frac{1}{\kappa}\right)^{-0.5} \Gamma\left(\frac{3}{\kappa}\right)^{-0.5}.$$

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Here, κ is the shape parameter with constraint $\kappa > 0$, λ is skewness parameter with $-1 < \lambda < 1$. The SGED turns out to be the standard normal distribution when $\kappa = 2$ and $\lambda = 0$. The log-likelihood function of GARCH-SGED model is

$$L(\psi) = -\frac{|R_t - \mu / \sigma_t + \delta|^\kappa}{[1 + \text{sign}(R_t - \mu / \sigma_t + \delta)\lambda]^\kappa \theta^\kappa}$$

where: ψ is the parameter vector. The one-day-ahead VaR forecast based on SGED is defined as

$$VaR_{t+1} = \mu_{t+1} + \phi_p^{-1}(\varepsilon_t) \sigma_{t+1}$$

where: $\phi_p^{-1}(\varepsilon_t)$ is the left quantile of SGED at p level.

Now, we show that GARCH models with the EOLLN distribution yields quite satisfactory results and compare our proposed model with GARCH models where normal distribution, Student's-t distribution, GED and SGED are considered as innovation distributions. For this aim, we propose the novel VaR model, called as GARCH-EOLLN, to forecast daily VaR.

The proposed dynamic VaR model can be briefly summarized as follows: At the first stage, GARCH model is fitted to financial return series to obtain one-day-ahead forecast of conditional mean variance and extract standardized residuals for the second stage by the pseudo maximum likelihood method. At the second stage, the EOLLN distribution is used to model extracted standardized residuals from stage one to obtain quantile estimation for given confidence level. Finally, the VaR is estimated using the quantile estimation of the EOLLN and one-day-ahead forecast of conditional mean and variance. The VaR forecasting performance of the proposed model is compared with the GARCH models specified under skewed and heavy-tailed distributions by means of backtesting procedure. In summary, proposed model has the following steps:

- **Step 1.** The benchmark model, GARCH (1,1) is fitted to the return series by the pseudo maximum likelihood estimation (PML) to maximize the log-likelihood function assuming normal innovations and obtain the one day ahead forecasts of μ_{t+1} and σ_{t+1} from the fitted model and extract the standardized residuals ε_t .
- **Step 2.** Standardized residual obtained from the first step is modeled via EOLLN to estimate the quantile of innovation distribution at p level. Using the parameter estimation of EOLLN distribution and forecasts of μ_{t+1} and σ_{t+1} , VaR_{t+1} can be forecasted where:

$$VaR_{t+1} = \mu_{t+1} + \phi_p^{-1}(\varepsilon_t; a, b, \mu, \sigma) \sigma_{t+1}$$

Here, $\phi_p^{-1}(\varepsilon_t; a, b, \mu, \sigma^2)$ is the quantile of the EOLLN distribution at p level. One can easily estimate the quantile of EOLLN as follows:

$$\phi_p^{-1}(\varepsilon_t; a, b, \mu, \sigma) = \phi^{-1} \left\{ \frac{p^{\frac{1}{ab}}}{p^{\frac{1}{ab}} + (1 - p^{\frac{1}{b}})^a} \right\}$$

where: $\phi^{-1}(\cdot)$ is the quantile function for the normal distribution with location parameter μ and scale parameter σ .

IV. Evaluating VaR Models

Backtesting methodology is used to compare the forecasting accuracy of the GARCH models in terms of VaR forecasts. The backtesting methodology consists of comparing the VaR forecast with actual realized loss in the out-of-sample period. Kupiec (1995) proposed a likelihood ratio test of unconditional coverage (LR_{uc}) to evaluate the model accuracy. The LR test examines whether the failure rate is equal to expected one. LR_{uc} test statistic is given by

$$LR_{uc} = -2 \ln \left[\frac{p^{n_1} (1-p)^{n_0}}{\hat{\pi}^{n_1} (1-\hat{\pi})^{n_0}} \right] \square \chi_1^2$$

where: $\hat{\pi} = n_1 / (n_0 + n_1)$ is the maximum likelihood estimation of p , n_1 represents the total violations (violation occurs when the realized return exceeds the forecasted VaR value) and n_0 represents the total non-violations forecasts. Under the null hypothesis $H_0: p = \hat{\pi}$, the LR statistics follows a chi-square distribution with one degree of freedom.

Christoffersen (1998) proposed a likelihood ratio test of conditional coverage (LR_{cc}) to remove the lack of Kupiec's test. LR_{cc} test investigates both equality of failure rate and expected one and also independently distributed violations. LR_{cc} test statistic, under the null hypothesis that the failures are independent and equal to expected one, can be given as follows:

$$LR_{cc} = -2 \ln \left[\frac{(1-p)^{n_0} p^{n_1}}{(1-\pi_{01})^{n_{01}} \pi_{01}^{n_{00}} (1-\pi_{11})^{n_{11}} \pi_{11}^{n_{10}}} \right] \square \chi_2^2$$

where: n_{ij} is the number of observations with value i followed by j for $i, j = 0, 1$ and $\pi_{ij} = \frac{n_{ij}}{\sum_j n_{ij}}$ are the probabilities, $i, j = 1$ denotes the violation has been occurred otherwise

indicates the opposite case. The main advantage LR_{cc} of test is that it can reject a VaR model that generates too many or too few clustered violations (Marimoutou et al., 2009). LR_{cc} test statistics follows a chi-square distribution with two degrees of freedom.

V. Empirical Findings

Data

To evaluate the performance of the proposed approach, the NASDAQ -100 and S&P-500 indexes are used. The used time series data sets, for NASDAQ-100 and S&P-500 indexes, contain 1533 and 1278 daily observations from 03.08.2010 to 01.09.2016 and from

29.09.2012 to 27.09.2017, respectively. The descriptive statistics of NASDAQ -100 and S&P-500 indexes are given in Table 2.

Table 2
Summary Statistics for the NASDAQ -100 and S&P-500 Indexes

Descriptive Statistics	NASDAQ -100	S&P-500
Number of observations	1533	1278
Minimum	-0.063	-0.0402
Maximum	0.0493	0.0383
Mean	0.0006	0.0005
Median	0.0008	0.0004
Std. Deviation	0.0107	0.0077
Skewness	-0.354	-0.3725
Kurtosis	3.866	5.5761
Jarque-Bera	559.913 (0.000)	382.921 (0.000)
Ljung-Box	49.425 (0.014)	23.878 (0.2478)

Table 2 shows that the mean returns are closed to 0. The Jarque–Bera statistics in Table 2 also show that the null hypothesis of normality is rejected at any level of significance for both indexes, as evidenced by high excess kurtosis and negative skewness. Thus, it is clear that log returns of NASDAQ-100 and S&P-500 indexes have non-normal characteristics, excess kurtosis, and fat tails.

Table 3 shows the estimated parameters of the GARCH(1,1) model. As seen from Table 3, the constant term, ARCH(1), and GARCH(1) parameters are highly significant for both indexes.

Table 3
GARCH(1,1) Parameter Estimation for the NASDAQ -100 and S&P-500 Indexes

NASDAQ-100				S&P-500		
Parameters	Constant	ARCH(1)	GARCH(1)	Constant	ARCH(1)	GARCH(1)
Estimates	0.00000495	0.1074	0.8467	0.000006331	0.1748	0.7165
SE	0.00000136	0.0254	0.0295	0.000001843	0.03846	0.05345
t-value	3.650	4.23	28.716	3.434	4.543	13.406
p-value	0.000262	0.0000233	<0.00001	0.000594	<0.00001	<0.00001

Table 4 shows the diagnostic statistics for raw data and standardized residuals extracted from GARCH(1,1) model. The residual series have significant excess skewness-kurtosis and normal distribution assumption is not realistic and also LM-test result indicates the ARCH effects.

Table 4
Diagnostic Statistics of Raw Data and Standardized Residuals of GARCH(1,1)

NASDAQ-100			S&P-500	
Series	Return series	Residuals series	Return series	Residuals series
Jarque-Bera	559.913 (0.000)	59.362 (0.000)	382.921 (0.000)	212.292 (0.000)
Ljung-Box	49.425 (0.014)	7.429 (0.684)	23.878 (0.2478)	23.128 (0.2826)
LM-test	258.881 (0.000)	9.9655 (0.619)	198.0501 (0.000)	10.8599 (0.9498)

Extracted residual from GARCH(1,1) process is modeled with normal, GED, SGED, and the proposed distribution. The parameter estimations are given in Table 5. Based on the figures in Table 5, the proposed distribution provides overall best fit and the best explanation of the standardized residual. Besides, the proposed distribution has the lowest log-likelihood value among all of other alternative models. Therefore, this model could be the best for the NASDAQ -100 and S&P-500 indexes.

Table 5

Estimated Parameters of Distributions

Index	Models	Parameter Estimates	$-\ell$
NASDAQ Q-100	Normal	$\mu = -0.041, \sigma = 0.999$	1607.595
	GED	$\mu = -0.011, \lambda = 1.000, \kappa = 1.352$	1586.710
	SGED	$\mu = -0.044, \lambda = 0.998, \kappa = 1.378, \theta = 0.900$	1584.523
	Proposed Distribution-EOLLN	$\alpha = 3.772, \beta = 0.687, \mu = 0.273, \sigma = 2.875$	1582.972
S&P-500	Normal	$\mu = 0.0515, \sigma = 0.999$	1812.163
	GED	$\mu = 0.067, \lambda = 0.9992, \kappa = 1.214$	1769.584
	SGED	$\mu = 0.0447170, \lambda = 0.995, \kappa = 1.248, \theta = 0.9$	1768.258
	Proposed Distribution-EOLLN	$\alpha = 0.483, \beta = 0.128, \mu = 1.459, \sigma = 0.327$	1760.519

* Parameter estimations are obtained for non-standardized GED and SGED.

Dynamic VaR Estimation with Rolling Window

In this subsection, GARCH(1,1) model is estimated using a rolling window procedure. Rolling window estimation procedure allows us to capture time-varying characteristics of the time series in different time periods. Window lengths are determined as 1133 and 878, respectively. The next 400 daily returns are used to evaluate the out of sample performance of VaR models.

Empirical Results of NASDAQ-100 Index

NASDAQ-100 index is used to evaluate the out of sample performance of VaR models. All used VaR models are evaluated by comparing the actual and expected failure rates and using two backtesting tests.

Table 6 shows the backtesting results of GARCH(1,1)-normal, GARCH(1,1)-Student's-t, GARCH(1,1)-GED, GARCH(1,1)-SGED and GARCH-EOLLN models for left tail of loss distribution. According to failure rates, GARCH models specified under the normal, Student's-t, GED and SGED innovation distributions perform poorly for both confidence levels; whereas, the proposed model outperforms to other models and shows great consistency for both confidence levels. Note that only the proposed model has equal expected and actual failure rates for 0.95 and 0.99 confidence levels.

Backtesting results based on the left tail of loss distribution show that GARCH(1,1)-normal, GARCH(1,1)-Student's-t and GARCH(1,1)-GED models underperform and rejected for both backtesting at $p=0.01$ level. Although GARCH(1,1)-SGED model performs well at both confidence levels according to backtesting results, one can easily see from Table 6 that GARCH(1,1)-SGED model causes the overestimated VaR forecasts.

Table 6

Out-of-sample Performance of VaR Models by Means of Backtesting Results for the Left Tail of Loss Distribution ($p=0.05$ and $p=0.01$)

p=0.05						
Nasdaq-100	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LRuc	LRcc
GARCH-Normal	400	20	27	6.8%	2.335 (0.126)	2.354 (0.308)
GARCH-Student's t	400	20	28	7.0%	3.012 (0.083)	3.013 (0.222)
GARCH-GED	400	20	26	6.5%	1.738 (0.187)	1.798 (0.407)
GARCH-SGED	400	20	25	6.3%	1.223 (0.269)	1.350 (0.509)
Proposed Model	400	20	20	5.0%	0.000 (1)	0.599 (0.645)
p=0.01						
Nasdaq-100	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LRuc	LRcc
GARCH-Normal	400	4	10	2.5%	6.417 (0.011)	7.806 (0.02)
GARCH-Student's t	400	4	9	2.3%	4.660 (0.031)	6.406 (0.041)
GARCH-GED	400	4	9	2.3%	4.660 (0.031)	6.406 (0.041)
GARCH-SGED	400	4	8	2.0%	3.131 (0.077)	5.300 (0.071)
Proposed Approach	400	4	4	1.0%	0.000 (1)	0.177 (1)

*The values in parenthesis represent the p values of LRuc and LRcc tests.

Table 7

Out-of-sample Performance of VaR Models by Means of Backtesting Results for the Right Tail of Loss Distribution ($p=0.95$ and $p=0.99$)

p=0.95						
Nasdaq-100	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LR-uc	LR-cc
GARCH-Normal	400	20	15	3.8%	1.485 (0.231)	1.736 (0.420)
GARCH-Student's t	400	20	15	3.8%	1.485 (0.231)	1.736 (0.420)
GARCH-GED	400	20	13	3.3%	2.928 (0.087)	3.548 (0.170)
GARCH-SGED	400	20	19	4.8%	0.053 (0.817)	0.064 (0.968)

p=0.95						
Nasdaq-100	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LR-uc	LR-cc
Proposed Model	400	20	19	4.8%	0.053 (0.817)	0.064 (0.968)
p=0.99						
Nasdaq-100	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LR-uc	LR-cc
GARCH-Normal	400	4	2	0.5%	1.237 (0.266)	1.257 (0.533)
GARCH-Student's t	400	4	1	0.3%	3.250 (0.071)	3.255 (0.196)
GARCH-GED	400	4	1	0.3%	3.250 (0.071)	3.255 (0.196)
GARCH-SGED	400	4	2	0.5%	1.237 (0.266)	1.257 (0.533)
Proposed Model	400	4	3	0.8%	0.276 (0.599)	0.322 (0.851)

*The values in parenthesis represent the p values of LR-uc and LR-cc tests.

Table 7 shows the backtesting results of VaR models for right tail of loss distribution. The all used VaR models perform well according to two backtesting results. However, when the observed failure rates of VAR models are examined, it is clear that proposed approach and GARCH-SGED models are closer to expected failure rate than other models. Proposed model and GARCH-SGED models show similar performance for right tail of loss distribution. In summary, GARCH-EOLLN performs the best according to both backtesting results and failure rate values for both tail of loss distribution.

Empirical Results of S&P-500 Index

In this subsection, S&P-500 index is used. Table 8 shows the backtesting results of VaR models for left tail of loss distribution. Based on the figures in Table 8, it is clear that proposed model outperforms among others. According to backtesting results, all models perform well, except GARCH(1,1)-Normal and GARCH-Student's t for p=0.01 level. Considering the failure rates, GARCH models specified under the normal, Student's-t, GED and SGED innovation distributions perform poorly for both confidence levels. Based on the figures in Table 8, it is clear that proposed model outperforms among others and failure rates of proposed distribution are closed to nominal values.

Table 8
Out-of-sample Performance of VaR Models by Means of Backtesting Results for the Left Tail of Loss Distribution (p=0.05 and p=0.01)

p=0.05						
S&P-500	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LR-uc	LR-cc
GARCH-Normal	400	20	13	3.3%	2.928 (0.087)	2.533 (0.315)
GARCH-Student's t	400	20	14	3.5%	2.107 (0.147)	3.548 (0.170)

Forecasting Value-at-Risk with Two-Step Method

p=0.05						
S&P-500	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LR-uc	LR-cc
GARCH-GED	400	20	13	3.3%	2.928 (0.087)	2.533 (0.315)
GARCH-SGED	400	20	12	3.0%	1.860 (0.158)	1.877 (0.402)
Proposed Model	400	20	20	5.0%	0.000 (1)	0.599 (0.645)
p=0.01						
S&P-500	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LR-uc	LR-cc
GARCH-Normal	400	4	9	2.3%	4.660 (0.031)	6.406 (0.041)
GARCH-Student's t	400	4	9	2.3%	4.660 (0.031)	6.406 (0.041)
GARCH-GED	400	4	8	2.0%	3.131 (0.077)	5.300 (0.071)
GARCH-SGED	400	4	8	2.0%	3.131 (0.077)	5.300 (0.071)
Proposed Approach	400	4	3	0.8%	0.276 (0.599)	0.322 (0.851)

*The values in parenthesis represent the p values of LR-uc and Lr-cc tests.

Table 9 shows the backtesting results of VaR models for right tail of loss distribution. The obtained results are similar to results of right tail of loss distribution. The proposed distribution shows great consistency for both tail of loss distribution. It is strong evidence that two-step GARCH-EOLLN model produce high accuracy VaR forecasts.

Table 9
Out-of-sample Performance of VaR Models by Means of Backtesting Results for the Right Tail of Loss Distribution (p=0.95 and p=0.99)

p=0.95						
S&P-500	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LR-uc	LR-cc
GARCH-Normal	400	20	10	2.5%	6.398 (0.011)	2.145 (0.252)
GARCH-Student's t	400	20	11	2.8%	5.059 (0.024)	1.540 (0.340)
GARCH-GED	400	20	10	2.5%	6.398 (0.011)	2.145 (0.252)
GARCH-SGED	400	20	12	3.0%	3.907 (0.048)	1.115 (0.454)
Proposed Model	400	20	20	5.0%	0.000 (1)	0.599 (0.645)

p=0.99						
S&P-500	Number of Forecasts	Expected Violation	Observed Violation	Failure Rate	LR-uc	LR-cc
GARCH-Normal	400	4	3	0.8%	0.276 (0.599)	0.322 (0.851)
GARCH-Student's t	400	4	3	0.8%	0.276 (0.599)	0.322 (0.851)
GARCH-GED	400	4	2	0.5%	1.237 (0.266)	1.257 (0.533)
GARCH-SGED	400	4	3	0.8%	0.276 (0.599)	0.322 (0.851)
Proposed Model	400	4	4	1.0%	0.000 (1)	0.177 (1)

*The values in parenthesis represent the p values of LR-uc and LR-cc tests.

VI. Conclusion

This study assesses the performance of proposed distribution, called as “exponentiated odd log-logistic normal distribution” (for short EOLLN), to forecast the VaR by means of two-step method and improves the forecasting accuracy at high quantiles for both tail of loss distribution. Proposed VaR model are compared with conventional models: GARCH-normal, GARCH-Student’s-t, GARCH-generalized error distribution and GARCH-skewed generalized error distribution by means of backtesting. According to backtesting results, GARCH-EOLLN model outperforms to other models in view of accuracy of VaR forecasts. The contribution of this study can be summarized as follows: EOLLN distribution is applied to VaR methodology successfully. The proposed two-step method has proved its ability to capture leptokurtic features of loss distribution. EOLLN distribution provides great fit to the left tail of loss distribution which contains extreme and unpredictable losses. It is clear that the proposed two-step model can be used to forecast market VaR value by financial institutions and regulators.

References

- Angelidis, T., Benos, A. and Degiannakis, S., 2004. The use of GARCH models in VaR estimation. *Statistical methodology*, 1(1), pp.105-128.
- Braga, A.S. Cordeiro, G.M. Ortega, E.M.M. and Cruz, J.N., 2016. The odd log–logistic normal distribution: Theory and applications in analysis of experiments. *Journal of Statistical Theory and Practice*, 10(2), pp.311-335.
- Bollerslev, T., 1986. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31(3), pp.307-327.
- Bollerslev, T., 1987. A conditional heteroskedastic time series model for speculative prices and rates of return. *The Review of Econometrics and Statistics*, 69(3), pp.542-547.
- Braione, M., and Scholtes, N. K., 2016. Forecasting Value-at-Risk under Different Distributional Assumptions. *Econometrics*, 4(3), pp.1-27.
- Engle, R.F., 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica: Journal of the Econometric Society*, pp. 987-1007.

- Exton, H., 1978. *Handbook of hypergeometric integrals: Theory, applications, tables, computer programs*. New York, NY: Halsted Press.
- Gradshteyn, I.S. and Ryzhik, I.M., 2007. *Table of integrals, series, and products*. San Diego, CA: Academic Press.
- Harmantzis, F.C. Miao, L. and Chien, Y., 2006. Empirical study of value-at-risk and expected shortfall models with heavy tails. *The Journal of Risk Finance*, 7(2), pp.117-15.
- Hung, J.C. Lee, M.C. and Liu, H. C., 2008. Estimation of value-at-risk for energy commodities via fat-tailed GARCH models. *Energy Economics*, 30(3), pp.1173-1191.
- Kuester, K., Mittnik, S. and Paolella, M. S., 2006. Value-at-risk prediction: A comparison of alternative strategies. *Journal of Financial Econometrics*, 4, pp.53–89.
- Kupiec, P.H., 1995. Techniques for verifying the accuracy of risk measurement models. *The Journal of Derivatives*, 3(2), pp.73-84.
- Lee, M.C. Su, J.B. and Liu, H.C., 2008. Value-at-risk in US stock indices with skewed generalized error distribution. *Applied Financial Economics Letters*, 4(6), pp.425-431.
- Marimoutou, V. Raggad, B. and Trabelsi, A., 2009. Extreme value theory and value at risk: application to oil market. *Energy Economics*, 31(4), pp.519-530.
- McMillan, D.G. and Kambouroudis, D., 2009. Are RiskMetrics forecasts good enough? Evidence from 31 stock markets. *International Review of Financial Analysis*, 18(3), pp.117–124.
- Nadarajah, S., 2008. Explicit expressions for moments of order statistics. *Statistics & Probability Letters*, 78, pp.196–205.
- Nelson, D., 1991. Conditional heteroscedasticity in asset returns: a new approach. *Econometrica*, 59 pp.347-370.
- Prudnikov, A.P. Brychkov, Y.A. and Marichev, O.I., 1986. *Integrals and series: Elementary functions*. New York, NY: Gordon and Breach Science Publishers.
- Venkataraman, S., 1997. Value at risk for a mixture of normal distributions: the use of quasi-Bayesian estimation techniques. *Economic Perspectives-Federal Reserve Bank of Chicago*, 21, pp.2-13.
- Zangari, P., 1996. An improved methodology for measuring VaR. *RiskMetrics Monitor*, 2(1), pp.7-25.

Appendix

Expansions

In this subsection, we provide alternative mixture representations for the pdf and cdf of X. Despite the fact that the pdf and cdf of EOLLN require mathematical functions that are widely available in modern statistical packages, frequently analytical and numerical derivations take advantage of power series for the pdf. Some useful expansions for (2) can be derived by using the concept of power series. First, we obtain an expansion for the cdf of EOLLN using

a power series for $\Phi\left(\frac{X-\mu}{\sigma}\right)^{\alpha\beta}$

$$\begin{aligned} \Phi\left(\frac{X-\mu}{\sigma}\right)^{\alpha\beta} &= \sum_{i=0}^{\infty} (-1)^i \binom{\alpha\beta}{i} \left(1 - \Phi\left(\frac{X-\mu}{\sigma}\right)\right)^i \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{i+k} \binom{\alpha\beta}{i} \binom{i}{k} \Phi\left(\frac{X-\mu}{\sigma}\right)^k \end{aligned}$$

where: $\alpha, \beta > 0$. Then, we have

$$\Phi\left(\frac{X-\mu}{\sigma}\right)^{\alpha\beta} = \sum_{k=0}^{\infty} a_k \Phi\left(\frac{X-\mu}{\sigma}\right)^k$$

Here, $a_k = \sum_{i=k}^{\infty} (-1)^{i+k} \binom{\alpha\beta}{i} \binom{i}{k}$ and $\{\Phi\left(\frac{X-\mu}{\sigma}\right)^{\alpha} + [1 - \Phi\left(\frac{X-\mu}{\sigma}\right)]^{\beta}\} = \sum_{k=0}^{\infty} b_k \Phi\left(\frac{X-\mu}{\sigma}\right)^k$

where: $b_k = h_k(\alpha, \beta)$. Then, using the ratio of two power series, we obtain

$$F(x) = \frac{\sum_{k=0}^{\infty} a_k \Phi\left(\frac{X-\mu}{\sigma}\right)^k}{\sum_{k=0}^{\infty} b_k \Phi\left(\frac{X-\mu}{\sigma}\right)^k} = \sum_{k=0}^{\infty} c_k \Phi\left(\frac{X-\mu}{\sigma}\right)^k \tag{4}$$

where: $c_0 = \frac{a_0}{b_0}$ and the coefficients c_k for $k \geq 1$ are determined from the recurrence equation

$$c_k = \frac{1}{b_0} \left[a_k - \frac{1}{b_0} \sum_{r=1}^k b_r c_{k-r} \right]$$

The pdf of X is obtained by differentiating Equation (4) as

$$f(x) = \sum_{k=0}^{\infty} \frac{c_{k+1}(k+1)}{\sigma} \phi\left(\frac{X-\mu}{\sigma}\right) \Phi\left(\frac{X-\mu}{\sigma}\right)^k \tag{5}$$

Equation (5) reveals that the EOLLN density function is a mixture of Exp-N densities. Thus, some of its structural properties such as the ordinary and incomplete moments and generating function can be obtained from well-established properties of the Exp-N distribution. This equation is the main result of this section.

Moments and Moment Generating Function

Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis). Now, we obtain ordinary and incomplete moments of the EOLLN distribution. Henceforth, let $Z = (X - \mu)/\sigma$ be a random error variable, where X has density function given by (2). Then, the random variable Z has the EOLLN(0, 1, α , β) distribution. The moments of X having the EOLLN (μ , σ , α , β) distribution are easily determined from the moments of Z by

$$E(X^n) = E[(\mu + \sigma Z)^n] = \sum_{r=0}^n \binom{n}{r} \mu^{n-r} \sigma^r E(Z^r)$$

Thus, we can work with the standardized random variable Z . The first representation for μ'_n is based on the (n, r) th probability weighted moment (PWM) (for n and r positive integers) of the standard normal distribution defined by,

$$\tau_{n,r} = \int_{-\infty}^{\infty} x^n \Phi^r(x) \phi(x) dx$$

We define the Lauricella function of type A (Exton, 1978) as

$$F_A^{(n)}(a, b_1, \dots, b_n; c_1, \dots, c_n; x_1, \dots, x_n) = \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\dots+m_n} (b_1)_{m_1} \dots (b_n)_{m_n}}{(c_1)_{m_1} \dots (c_n)_{m_n}} \frac{x_1^{m_1} \dots x_n^{m_n}}{m_1! m_2! \dots m_n!}$$

where: $(a)_k = a(a-1)\dots(a-k+1)$ denotes the Pochhammer symbol, i.e. the k th rising factorial power of a with the convention $(a)_0 = 1$. Nadarajah (2008) expressed $\tau_{n,r}$ in terms of the Lauricella function of type A as

$$\begin{aligned} \tau_{n,r} &= 2^{n/2} \pi^{-(r+\frac{1}{2})} \sum_{\substack{p=0 \\ (n+r-p)\text{even}}}^r \binom{r}{p} (\pi/2)^p \Gamma\left(\frac{n+r-p+1}{2}\right) \\ &\times F_A^{(r-p)}\left(\frac{n+r-p+1}{2}; \frac{1}{2}, \dots, \frac{1}{2}, \frac{3}{2}, \dots, \frac{3}{2}; -1, \dots, -1\right) \end{aligned} \tag{6}$$

The n th ordinary moment of Z is given by

$$\mu'_n = \sum_{k=0}^{\infty} (k+1)c_{k+1} \int_{-\infty}^{\infty} x^n \Phi(x)^k \phi(x) dx = \sum_{k=0}^{\infty} (k+1)c_{k+1} \tau_{n,k}$$

where: $\tau_{n,k}$ can be determined from Equation (6)