Associate Professor Ruxing Xu, PhD
E-mail: xrxing@cjlu.edu.cn
Associate Professor Dan Wu, PhD
E-mail: wudan@cjlu.edu.cn
Department of Mathematics, China Jiliang University
Professor Ronghua Yi, PhD
E-mail: yirh@cjlu.edu.cn
School of Management, China Jiliang University

PRICING CDSS AND CDS OPTIONS UNDER A REGIME-SWITCHING CEV PROCESS WITH JUMP TO DEFAULT

Abstract. *This paper studies the valuation of credit default swaps (CDSs) and CDS options of European and Bermudan styles under a regime-switching constant elasticity of variance (CEV) process with jump to default. Based on the empirical evidence that the changes of macroeconomic conditions such as business cycle impact on the values of credit products, we assume that the interest rate, the volatility parameter of the CEV process, and the parameters of default intensity function have switching dynamics governed by a continuous-time finite state Markov chain, whose states are deemed to represent the states of the underlying economy. We construct a recombining trinomial lattice and demonstrate the accuracy of the lattice framework. Within the framework, we derive the values of CDSs, European and Bermudan CDS options. The numerical results provide insight into the impact of regime switching on the behavior of CDS spread rates and the values of European and Bermudan CDS options.*

Keyword: *Credit default swap, CDS options, CEV process, Lattice model, Regime switching*

JEL Classification: C63; G13

1. Introduction

In recent years, we have witnessed an explosive growth in the credit derivative market. By far CDSs and CDS options are the most popular credit derivatives traded in credit markets. In a CDS contract, the protection buyer pays a periodic premium, called CDS spread, to the protection seller until the maturity of the contract or a default event of a reference entity, whichever is earlier. Upon the occurrence of the default event, the buyer receives from the seller the difference between the par value and the recovery value of the reference entity as compensation. A forward CDS contract is the obligation to buy or sell a CDS on a specific reference entity for a specified spread at a specified future time. A CDS option is an option that gives its holder the right, but not the obligation, to enter into a CDS with a specified spread before or at option expiration date.

There are several recent academic studies on the valuation of CDSs and CDS options. Hull and White (2000, 2001) provide a methodology for valuing CDSs when the payoff is contingent on default by a single reference entity with and without counterparty risk. Hull and White (2003) explain the derivation of the Black-Scholes pricing formula for European CDS options. Campi *et al*. (2009) value CDSs in a model where the CEV process is killed at the first jump time of an independent Poisson process with constant intensity. However, this assumption is not consistent with the empirical evidence that indicates a close relation between the default probability and the equity price, as well as the equity volatility. To overcome this shortcoming, Carr and Linetsky (2006) generalize the reduced-form approach to include a process for equity, and take the equity price to follow a CEV diffusion, punctuated by a possible jump to zero. To capture the possible positive link between default and volatility, they specify the default intensity as an affine function of the instantaneous variance of the underlying equity. Thus, the CEV model extended with jump to default demonstrates both the volatility skew and the relation between default probability and equity price as well as equity volatility. Following the model setup of Carr and Linetsky (2006), Mendoza-Arriaga and Linetsky (2011) develop an analytical solution to the equity default swaps pricing problem under the jump-to-default extended CEV.

However, there are substantial empirical evidences in support of the existence of regime switching effects on stock market returns and default probabilities. Using the CRSP stock market returns over the period 1929-1989, Schaller and Norden (1997) demonstrate that there is compelling evidence of regime switching in US stock market returns and the evidence for switching is robust to different specifications such as switching in means, switching in variances, and switching in both means and variances. Based on a database of issuer-level default probabilities covering the period 1987-2000, Das *et al*. (2006) observe that default probabilities vary with the state of the economy and find strong support for regime-dependent default intensity model. Alexander and Kaeck (2008) also find empirical evidence that determinants of credit spreads are regime-dependent. Ang and Timmermann (2012) show that regime-switching models can capture the stylized behavior of many asset returns such as fat tails, heteroskedasticity, and skewness. All these results show that it is necessary to react the effects of regime switching in modelling equity price and default risk.

In this article, we extend the work of Carr and Linetsky (2006) by allowing for a switching regime structure in the dynamics of equity price and default intensity, and propose a lattice-based approach for pricing CDSs, European and Bermudan CDS options. The intuition behind the regime-switching model given here is to incorporate the impact of the changes of macroeconomic condition such as business cycle on the CDSs and CDS options. There is substantial empirical evidence supporting the existence of the switching behavior of interest rate. For example, Ang and Bekaert (2002) provide empirical evidence supporting the presence of regime switches and show that the regimes of interest rate might be ascribed to business cycle. Yao *et al.* (2006, p.281) mention that it is important to allow the model parameters to react to the market movement since the trend of the market is a key factor that governs the movement of equity price. The regime-switching version of the CEV model with jump to default could provide a more realistic way to react random market environment. Under the regime-switching CEV model with jump to default, we derive the values of CDSs and CDS options The results provide insight into the impact of regime switching on the behavior of CDS spread rates and CDS options.

The rest of this paper is organized as follows. Section 2 develops a regime-switching CEV model with jump to default. Section 3 constructs a trinomial lattice for the regime-switching CEV process and shows the convergence of the lattice. Section 4 shows how to use the lattice method to determine the valuation of CDSs and forward CDSs. Section 5 presents the pricing of European and Bermudan CDS options based on the lattice method. Section 6 presents the numerical results that explain the impact of regime switching and model parameters on the spread rates and the values of the CDS options. Conclusions are given in Section 7.

2. Regime-switching CEV model with jump to default

In this section we present the price dynamics under the regime-switching CEV model with jump to default. We start with a complete probability space (Ω, F, P), upon which all stochastic processes are defined.

We assume that the market is frictionless, there is no arbitrage and an equivalent martingale measure (EMM) P is given.

Let X_t be a continuous-time, G-state, Markov chain on (Ω, F, P), where G is the total number of states considered in the economy. Each state represents a particular regime and is labeled by an integer i between 1 and G. Hence the state space of X_t is given by $M := \{1, 2, \cdots, G\}$. X_t is assumed to be observable and serve as a proxy for some observable exogenous economic factors such as business cycle and stock price index.

To obtain the transition probabilities of the Markov chain X_t, we need to specify its generator matrix Q. For easy exposition, we assume that a constant generator $Q = (q_{ij})_{G\times G}$ is given, whose elements $(q_{ij})_{G\times G}$ satisfy:

1) $q_{ij} \geq 0$ if $i \neq j$;
2) $q_{ii} \leq 0$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$ for each $i = 1, \cdots, G$.

Assume that the Markov chain X_t at any time $t > 0$ is in regime $i \in M$. Then after a period of time Δt, $X_{t+\Delta t}$ may stay in regime i with probability $p_{i,i}^X$ or jump to any other regime $j \in M$ with probability $p_{i,j}^X$, where the one-step transition probabilities $p_{i,j}^X$ of the Markov chain X_t are given by

$$p_{i,j}^X = P\{X_{t+\Delta t} = j \mid X_t = i\} = \begin{cases} e^{q_{ii}\Delta t}, j = i, \\ (1 - e^{q_{ii}\Delta t})\frac{q_{ij}}{-q_{ii}}, j \neq i. \end{cases} \quad (1)$$

Independent of the Markov chain X_t, a standard Brownian motion W_t is defined on the probability space (Ω, F, P). We suppose that the pre-default equity price process S_t under the measure P evolves over time according to the following regime-switching CEV process:

$$dS_t = [r_{X_t} + \pi_{X_t}(S_t)]S_t dt + \sigma_{X_t} S_t^{\beta} dW_t, S_0 = s > 0, \quad (2)$$

where $0 < \beta < 1$ is the constant elasticity parameter. r_{X_t}, σ_{X_t}, and $\pi_{X_t}(S_t)$ are risk free interest rate, instantaneous volatility, and time- and equity-price-dependent default intensity respectively, all of which are modulated by the Markov chain X_t, indicating that they can take different values in different regimes, where we assume that r_i and σ_i are positive and $\pi_i(S_t)$ remains uniformly bounded as $S_t \to +\infty$ for each $i \in M$. The addition of the default intensity in the drift rate in the pre-default equity price process (2) compensates for the default jump to insure that the total expected rate of return to the equityholder is equal to the risk-free rate in the risk-neutral economy and the discounted gain process is a martingale under EMM.

Default is represented by the equity price dropping to zero and remaining thereafter frozen at that level, which can occur in one of two ways: the equity price S_t either hits zero via diffusion or jumps to zero from some positive value, whichever occurs first. Formally, we assume $\xi_1 = \inf\{t \geq 0 : S_t = 0\}$ is the first hitting time of zero for the diffusion process (2), and $\xi_2 = \inf\{t \geq 0 : \int_0^t \pi_{X_u}(S_u)du \geq e\}$ is the jump-to-default time with default intensity $\pi_{X_t}(S_t)$, i.e., the first jump time of a doubly stochastic Poisson process with intensity $\pi_{X_t}(S_t)$, where e is an exponential random variable with unit parameter and independent of X_t and W_t. Thus the time of default ξ can be given as $\xi = \min\{\xi_1, \xi_2\}$.

To capture the possible positive link between default intensity and equity volatility, we incorporate regime switching parameters in the default intensity and specify as

$$\pi_{X_t}(S_t) = a_{X_t} + b_{X_t}\sigma_{X_t}^2 S^{2(\beta-1)}, \tag{3}$$

where a_{X_t} and b_{X_t} are modulated by the Markov chain X_t and a_i and b_i are positive for each $i \in M$. Then the default probability $p^{X_t}(S_t)$ during the interval Δt is given

$$p^{X_t}(S_t) = 1 - e^{-\pi_{X_t}(S_t)\Delta t}. \tag{4}$$

3. Recombining trinomial lattice for the regime- switching CEV proess with jump to default

3.1. Construction of recombining trinomial lattice

We next present the details of our lattice construction for the regime-switching CEV diffusion with jump to default. Suppose at the current time the process (2) stays in regime $i \in M$. Since the process (2) has nonconstant volatility, following Nelson and Ramaswamy (1990), we use φ transform such that the stochastic process has constant volatility, which is given by

$$\varphi_t = \int_0^{S_t} \frac{1}{u^\beta} du = \frac{S_t^{1-\beta}}{1-\beta}. \tag{5}$$

By taking the inverse of the above equation, the equity price is given as

$$S_t = \left[(1-\beta)\varphi_t\right]^{\frac{1}{1-\beta}}. \tag{6}$$

Applying Ito's lemma to (5), the transformed equation becomes

$$d\varphi_t = \mu_i(\varphi_t)dt + \sigma_i dW_t, \tag{7}$$

where

$$\mu_i(\varphi_t) = (r_i + a_i)(1-\beta)\varphi_t + \frac{\sigma_i^2(2b_i - \beta)}{2(1-\beta)\varphi_t}. \tag{8}$$

We consider a continuous-time financial model where $T>0$ denotes the maturity of a reference entity such as a defaultable zero coupon bond. To construct the lattice, we divide the interval $[0,T]$ into N steps,

$$0,\ \Delta t,\ 2\Delta t,\ \cdots,\ (N-1)\Delta t,\ N\Delta t, \tag{9}$$

where $\Delta t=\frac{T}{N}$ is the size of time step. Because the φ process in different regime has different volatility, to keep the trinomial lattice as a recombined one with regime switching, we choose the same space step size $\Delta\varphi=\bar{\sigma}\sqrt{\Delta t}$ in all regimes, where $\bar{\sigma}$ is to be determined below. Then φ may be equal to

$$\varphi_0-M_{1i}\Delta\varphi,\ \cdots,\ \varphi_0-\Delta\varphi,\ \varphi_0,\ \varphi_0+\Delta\varphi,\ \cdots,\ \varphi_0+M_{2i}\Delta\varphi, \tag{10}$$

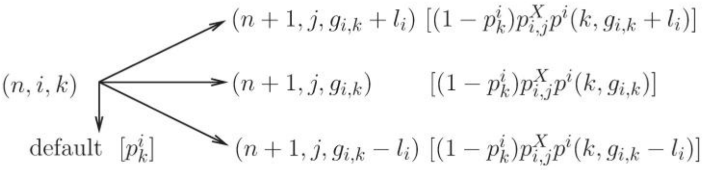
where M_{1i} and M_{2i} are regime-dependent positive integers. φ_0 is related to the initial equity price S_0 through the transform (5)

$$\varphi_0=\frac{S_0^{1-\beta}}{1-\beta}. \tag{11}$$

The following notations are used: t_n denotes $n\Delta t$, for $n=0,1,2,\cdots,N$, and φ_k denotes $\varphi_0+k\Delta\varphi$, for $k=0,\pm1,\pm2,\pm3,\cdots$. Then at time t_n, the node of the lattice can be denoted as (n,i,k): n for the time t_n, i for the regime i, and k for the transform φ_k. Assuming that default occures only at the nodes of the lattice and the regime does not change during the period $[t_n,t_{n+1})$.

We specify the trinomial branching process on this new three dimensional grid in the (t,X,φ)-space. The value of φ is restricted to move only to points on this grid, which ensures computational efficiency as the trinomial lattice will recombine properly. Suppose that the current node of the lattice is (n,i,k). We now show how this state evolves in the $(n+1)$ th step. First, $X_{t_{n+1}}$ may stay at the regime i with probability $p_{i,i}^X$ or jump to any other regime $j\neq i$ with probability $p_{i,j}^X$ which are defined in (1). Next, φ takes three different values depending on the values of $\mu_i(\varphi_k)$ and σ_i, resulting in three branches emanating from the current node. Since φ increases rapidly when φ is high or low enough according to Equation (7) and σ_i takes different value in different regime, at the $(n+1)$ th step φ takes three values: $\varphi_{g_{i,k}-l_i}$, $\varphi_{g_{i,k}}$, and $\varphi_{g_{i,k}+l_i}$, where $g_{i,k}$ is an integer depending on regime i and φ_k, and l_i is a positive integer depending on regime i, both of which will be discussed later in this section. Let $p^i(k,g_{i,k}-l_i)$, $p^i(k,g_{i,k})$, and $p^i(k,g_{i,k}+l_i)$ denote the probabilities of moving from φ_k to $\varphi_{g_{i,k}-l_i}$, $\varphi_{g_{i,k}}$, and $\varphi_{g_{i,k}+l_i}$ respectively conditional on regime i during time Δt. φ_k may jump to zero with the default probability $p^i(S_k)$ given in (1), here denoted by p_k^i for convenience, resulting in the fourth branch added to the node. Thus emanating from the node (n,i,k), for $j=1,2,\cdots,G$, there are $4G$ nodes with appropriate probabilities in square brackets shown in Figure 1.

Figure 1. The nodes emanating from the node (n,i,k)



To achieve consistency with the continuous-time dynamics (7), it is required that the mean and variance implied by this trinomial lattice to that implied by the diffusion should be matched, and that the branching probabilities add up to one:

$$p^i(k,g_{i,k}-l_i)(g_{i,k}-l_i-k)\Delta\varphi+p^i(k,g_{i,k})(g_{i,k}-k)\Delta\varphi+p^i(k,g_{i,k}+l_i)(g_{i,k}+l_i-k)\Delta\varphi=\mu_i(\varphi_k)\Delta t, \quad (12)$$

$$\begin{aligned}p^i(k,g_{i,k}-l_i)(g_{i,k}-l_i-k)^2\Delta\varphi^2+p^i(k,g_{i,k})(g_{i,k}-k)^2\Delta\varphi^2&\\ +p^i(k,g_{i,k}+l_i)(g_{i,k}+l_i-k)^2\Delta\varphi^2=\sigma_i^2\Delta t+\mu_i(\varphi_k)^2\Delta t^2,&\end{aligned} \quad (13)$$

$$p^i(k,g_{i,k}-l_i)+p^i(k,g_{i,k})+p^i(k,g_{i,k}+l_i)=1. \quad (14)$$

Solving the equations (12) to (14), the probabilities are given by

$$p^i(k,g_{i,k}-l_i)=\frac{1}{2}\left[\left(\frac{g_{i,k}-k}{l_i}-\frac{\mu_i(\varphi_k)\sqrt{\Delta t}}{l_i\bar{\sigma}}+\frac{1}{2}\right)^2+\frac{\sigma_i^2}{l_i^2\bar{\sigma}^2}-\frac{1}{4}\right], \quad (15)$$

$$p^i(k,g_{i,k})=1-\frac{\sigma_i^2}{l_i^2\bar{\sigma}^2}-\left(\frac{g_{i,k}-k}{l_i}-\frac{\mu_i(\varphi_k)\sqrt{\Delta t}}{l_i\bar{\sigma}}\right)^2, \quad (16)$$

$$p^i(k,g_{i,k}+l_i)=\frac{1}{2}\left[\left(\frac{g_{i,k}-k}{l_i}-\frac{\mu_i(\varphi_k)\sqrt{\Delta t}}{l_i\bar{\sigma}}-\frac{1}{2}\right)^2+\frac{\sigma_i^2}{l_i^2\bar{\sigma}^2}-\frac{1}{4}\right]. \quad (17)$$

It can be seen that the probabilities $p^i(k,g_{i,k}-l_i)$ and $p^i(k,g_{i,k}+l_i)$ are nonnegative if $\frac{1}{4}\le\frac{\sigma_i^2}{l_i^2\bar{\sigma}^2}$. Also, the probability $p^i(k,g_{i,k})$ must be nonnegative, which implies that $g_{i,k}$ should satisfy the following inequality:

$$k+\frac{\mu_i(\varphi_k)\sqrt{\Delta t}}{\bar{\sigma}}-\sqrt{l_i^2-\frac{\sigma_i^2}{\bar{\sigma}^2}}\le g_{i,k}\le k+\frac{\mu_i(\varphi_k)\sqrt{\Delta t}}{\bar{\sigma}}+\sqrt{l_i^2-\frac{\sigma_i^2}{\bar{\sigma}^2}}. \quad (18)$$

To ensure that some integer $g_{i,k}$ exists, the value of $l_i^2-\frac{\sigma_i^2}{\bar{\sigma}^2}$ should be more than or equal to $\frac{1}{4}$, or else no integer value may exist, which would satisfy (18). Hence, in order to have the positive probabilities, $\bar{\sigma}$ and l_i should satisfy the following constraint:

$$\frac{1}{4} \leq \frac{\sigma_i^2}{l_i^2 \bar{\sigma}^2} \leq 1 - \frac{1}{4l_i^2}. \tag{19}$$

It can be shown from (18) that $g_{i,k}$ may not be uniquely determined. Under such a condition, a simple solution is to choose $g_{i,k}$ such that the central branch of the trinomial tree should be placed as close as possible to the expected value of φ at the end of the period based on (7), i.e.,

$$g_{i,k} = \text{nint}\left(k + \frac{\mu_i(\varphi_k)\sqrt{\Delta t}}{\bar{\sigma}} \right), \tag{20}$$

where $\text{nint}(\cdot)$ is the nearest integer function which returns the integer that is closest to a real number.

3.2. Convergence of the lattice method for European call options

To demonstrate the accuracy of the lattice framework developed above, we price European call option with our lattice method, and numerically compare the results with the values calculated by Monte Carlo simulation based on the Euler-Maruyama discretization scheme. The terminal payoff of the option with strike price K and expiry date T is $(S_T - K)^+$, given no default by T, and is zero otherwise. Then at node (n,i,k) the value of the option $C_n(i,k)$ can be expressed in recursive form as

$$C_n(i,k) = e^{-r_i \Delta t}(1 - p_k^i) \sum_{j=1}^{G} p_{i,j}^X \sum_{m=-l_i,0,l_i} p^i(k, g_{i,k} + m) C_{n+1}(j, g_{i,k} + m). \tag{21}$$

For ease of illustration, we consider a two-regime Markov chain X_t with the generator matrix $Q = (q_{ij})_{2\times 2}$, where the elements are chosen as: $q_{12} = q_{21} = 0.6$. Other parameters are specified as: $S_0 = 100$, $\beta = 0.5$, $r_1 = 0.05$, $r_2 = 0.03$, $\sigma_1 = 0.3$, $\sigma_2 = 0.5$, $a_1 = 0.01$, $a_2 = 0.03$, $b_1 = 0.5$, $b_2 = 1$, the strike prices $K = 95$, 100 and 105, and their maturity $T = 1$ year.

Table 1 presents a comparison of European call prices generated by the lattice model and the Monte Carlo method. The numbers in the second column to the fourth column show results for assuming we start the first regime in regime 1, and in the fifth column to the seventh column assuming we start in regime 2. Ten different numbers for the time steps are used to illustrate the convergence of the lattice approximation. The results obtained from Monte Carlo simulation are reported in the last two rows in Table 1, where the numbers in the first row are the estimated option prices (EOP) and the numbers within parentheses are the standard deviations (SD), which are computed based on 100,000 simulation paths with 1,000 time steps.

Table 1. Convergence of European call options

Steps (N)	Regime 1			Regime 2		
	K=95	K=100	K=105	K=95	K=100	K=105
300	10.5922	5.9493	2.0211	10.7083	6.1896	2.5271
400	10.5969	5.9540	2.0249	10.7129	6.1941	2.5307
500	10.5995	5.9568	2.0271	10.7155	6.1967	2.5327
600	10.6015	5.9588	2.0286	10.7175	6.1986	2.5340
700	10.6030	5.9603	2.0297	10.7189	6.1999	2.5350
800	10.6036	5.9611	2.0303	10.7198	6.2008	2.5358
900	10.6046	5.9620	2.0310	10.7207	6.2261017	2.5364
1000	10.6052	5.9626	2.0315	10.7214	6.2023	2.5367
2000	10.6082	5.9656	2.0339	10.7237	6.2048	2.5388
3000	10.6088	5.9663	2.0346	10.7244	6.2055	2.5394
EOP	10.6030	5.9558	2.0434	10.7280	6.2183	2.5483
SD	0.012	0.011	0.008	0.015	0.014	0.018

For the European options, as shown in Table 1, the trinomial prices are stable and converge rapidly for the given strike prices as the number of time steps (N) increases. The prices generated by the lattice model are very close to those obtained by using the Monte-Carlo simulation, which supports the convergence of the lattice model.

4. Valuation of spot CDSs and forward CDSs

We consider a CDS that gives the holder the right to buy protection between times t_c and t_d ($0 \le t_c < t_d \le T$) on a defaultable zero-coupon bond with a face value of $L > 0$ and maturity T. When $t_c = 0$ this is a spot CDS, and when $t_c > 0$ it is a forward CDS. The protection buyer pays a premium rate of $U > 0$ per annum at times t_{c+1}, $\cdots$, t_{d-1}, and t_d in exchange for a protection payment at the default time ξ, provided that $t_c \le \xi \le t_d$. In practice, the premium rate U is also known as the CDS spread rate, which is chosen in such a way that the value of the CDS for the protection buyer, i.e., the present value of the protection leg minus the present value of the premium leg, is zero.

Now we price such a CDS with the lattice method proposed above. We first compute the value of the defaultable zero-coupon bond, denoted as $D_n(i,k)$ at any given node (n,i,k). At default time ξ, we assume that the recovery rate on the reference bond is expressed as an exogenous fraction γ ($0 < \gamma < 1$) of the market value of the bond just before default, which is proposed by Duffie and singleton (1999). Then at maturity T the bond's value $D_N(i,k)$ at node (N,i,k) is equal to γL when default happens or equal to L when there is no default. Thus we can get $D_n(i,k)$ by discounting the probability weighted average of all its children under the recovery of market value condition

$$D_n(i,k) = e^{-r_i\Delta t}(1-p_k^i+\gamma p_k^i)\sum_{j=1}^{G} p_{i,j}^X \sum_{m=-l_i,0,l_i} p^i(k, g_{i,k}+m) D_{n+1}(j, g_{i,k}+m). \tag{22}$$

Next we compute the expected value of the protection leg denoted as $\tilde{V}_n(i,k)$ at node (n,i,k) conditional on no default occurring. At the maturity date t_d, we have $\tilde{V}_d(i,k)=0$ if there is no default and $\tilde{V}_d(i,k)=(1-\lambda)D_d(i,k)$ if default happens at node (d,i,k). Then lattice-based recursive expression for $\tilde{V}_n(i,k)$ at node (n,i,k) during the protection period ($t_c \le t_n < t_d$) is as follows:

$$\tilde{V}_n(i,k) = e^{-r_i\Delta t}(1-p_k^i)\left\{\sum_{j=1}^{G} p_{i,j}^X \sum_{m=-l_i,0,l_i} p^i(k, g_{i,k}+m)\tilde{V}_{n+1}(j, g_{i,k}+m)\right\} + p_k^i(1-\gamma)D_n(i,k). \tag{23}$$

Since the forward CDS ceases to exist if the bond defaults before time t_c, the recursive expression for $\tilde{V}_n(i,k)$ ($t_n < t_c$) is given

$$\tilde{V}_n(i,k) = e^{-r_i\Delta t}(1-p_k^i)\sum_{j=1}^{G} p_{i,j}^X \sum_{m=-l_i,0,l_i} p^i(k, g_{i,k}+m)\tilde{V}_{n+1}(j, g_{i,k}+m). \tag{24}$$

Now we compute the value of \$1 payment per annum denoted as $\bar{V}_n(i,k)$ at node (n,i,k), conditional on no default occurring. At node (d,i,k) we have $\bar{V}_d(i,k)=0$. The lattice-based recursive expression for $\bar{V}_n(i,k)$ at node (n,i,k) is

$$\bar{V}_n(i,k) = e^{-r_i\Delta t}(1-p_k^i)\left\{\sum_{j=1}^{G} p_{i,j}^X \sum_{m=-l_i,0,l_i} p^i(k, g_{i,k}+m)\bar{V}_{n+1}(j, g_{i,k}+m) + \Delta t\right\}. \tag{25}$$

Because there is no premium payment before time t_{c+1}, the recursive expression for $\bar{V}_n(i,j)$ ($t_n < t_c$) is

$$\bar{V}_n(i,k) = e^{-r_i\Delta t}(1-p_k^i)\sum_{j=1}^{G} p_{i,j}^X \sum_{m=-l_i,0,l_i} p^i(k, g_{i,k}+m)\bar{V}_{n+1}(j, g_{i,k}+m). \tag{26}$$

The CDS spread is the specified spread that causes the contract to have a value of zero at inception. If the CDS contract is incepted at node (n,i,k), by equating the present value of the premium leg to the present value of the protection leg, the fairprice of CDS at that node can be given

$$U_n(i,k) = \frac{\tilde{V}_n(i,k)}{L\bar{V}_n(i,k)}. \tag{27}$$

Thus, the fair CDS spread at time zero with initial regime $X_0 = i$ can be computed as

$$U_0(i,0) = \frac{\tilde{V}_0(i,0)}{L\bar{V}_0(i,0)}. \tag{28}$$

5. Pricing European and Bermudan CDS options

European and Bermudan CDS options are traded actively in the Over-the-Counter (OTC) market. A European CDS option is an option on a CDS that gives its holder the right to enter into a CDS only at option maturity with a specified spread, i.e., the underlying CDS protection period starts at the option maturity. A Bermudan CDS option is an option on a CDS with a finite number of exercise opportunities. Note that a European CDS option is a particular Bermudan CDS option that only be exercised at the expiration date. Both types of options considered here are knocked out if the reference entity (the defaultable bond described above) defaults before the option maturity.

In this section we price European and Bermudan CDS options with maturity $0 < t_a \le t_d$ and strike spread $K > 0$ on the spot CDS in Section 2. For simplicity, the values of European and Bermudan CDS options are denoted as $V_n^e(i,k)$ and $V_n^b(i,k)$ at node (n,i,k) respectively.

5.1. Pricing a European CDS option

Now we focus on valuing a European CDS Option and then deal with its Bermuda counterpart in the next subsection. Suppose t_a is the maturity of a European CDS option with a specified spread K. Given that the bond does not default during the life of the option, the option's terminal value at node (a,i,k) is as follows:

$$V_a^e(i,k) = \max\left\{0, [U_a(i,k) - K] L \bar{V}_a(i,k)\right\}. \tag{29}$$

Then the lattice-based recursive expression for the option value at node (n,i,k) is

$$V_n^e(i,k) = e^{-r_i \Delta t}(1 - p_k^i) \sum_{j=1}^{G} p_{i,j}^X \sum_{m=-l_i,0,l_i} p^i(k, g_{i,k} + m) V_{n+1}^e(j, g_{i,k} + m). \tag{30}$$

Let $t_n = 0$, the value of European CDS option can be obtained.

5.2. Pricing a Bermudan CDS option

Different from the European CDS option above, a Bermudan CDS option is an option on a CDS with a finite number of exercise opportunities that are a subset of the CDS payment schedule t_1, t_2, $\cdots$, t_a.

At maturity date t_a, the value of the Bermudan CDS option is the same as that of the European CDS option, i.e., $V_a^b(i,k) = V_a^e(i,k)$ at node (a,i,k). If $t_n < t_a$ is an exercise date of the Bermudan option, the lattice-based recursive expression for the option value is

$$V_n^b(i,k)=\max\left\{g_n(i,k),(1-p_j^i)[U_n(i,k)-K]L\bar{V}_n(i,k)\right\}, \tag{31}$$

and if not, the lattice-based recursive expression for the option value is

$$V_n^b(i,k)=g_n(i,k), \tag{32}$$

where

$$g_n(i,k)=e^{-r_i\Delta t}(1-p_k^i)\sum_{j=1}^{G}p_{i,j}^X\sum_{m=-l_i,0,l_i}p^i(k,g_{i,k}+m)V_{n+1}^b(j,g_{i,k}+m) \tag{33}$$

is the holding value of the option at node (n,i,k).

Note that if the Bermudan CDS option can be exercised at CDS payment time t_1, t_2, $\cdots$, t_a, the option becomes American. Its value can be easily obtained with the above approach.

6. Numerical results

In this section we provide a numerical study of the effect of regime switching on the CDS spreads and the values of European and Bermudan CDS options. For ease of illustration, we consider a Markov chain with two regimes, where regime 1 represents a "good" economy and regime 2 represents a "bad" economy. We suppose that the elements of the generator matrix are chosen as: $q_{12}=q_{21}=\lambda>0$. Then the matrix of the Markov chain has the following form:

$$Q=\begin{pmatrix}-\lambda & \lambda\\ \lambda & -\lambda\end{pmatrix}, \tag{34}$$

which is also adopted in Boyle and Draviam (2007).

We consider two situations, i.e., one for the economy starting in regime 1 (the "good" regime) and the other for the economy starting in regime 2 (the "bad" regime). In this section, λ is set to 0, 0.2, 0.6, and 1.0. When $\lambda=0$, there isonly one regime, namely, the economy starting in regime 1 (2) will always stay in regime 1 (2). Since firms in the "good" regime is less likely to default than those in the "bad" regime, we choose $a_1=0.01$, $a_2=0.03$, $b_1=0.5$, and $b_2=1.0$. σ_1 and σ_2 are set to be 0.3 and 0.5. respectively, which is consistent with the fact that volatility is lower in "good" economy than in "bad" economy. Since higher interest rate leads to higher expected growth rate of the equity price according to (2), the interest rate in the "good" regime r_1 is set to be 0.05 and the interest rate in the "bad" regime r_2 is set to be 0.03, which ensures that the interest rate in the "good" economy is higher than that in the "bad" one. Other parameters for computation are as follows: $\Delta t=0.005$ (year), $S_0=100$, $L=100$, $\beta=0.5$, $\bar{\sigma}=0.1$, $\gamma=0.3$, $t_c=0$, $K=50$ bps, the maturities of defaultable zero-coupon bonds and CDSs $t_d=T=1$, 1.25, 1.5, $\cdots$, 4.5,4.75, 5 years, the maturities of

European and Bermudan CDS options $t_a = 1$ year. Bermudan CDS options'exercise dates are 0.25, 0.5, 0.75, 1 year.

Case 1: The impact of regime switching

Figure 2 depicts the spot CDS spread rates with maturities ranging from 1 to 5 years for varying value of the parameter λ. One can find that the spread rates implied by the regime-switching model are higher than those obtained in "good" regime in one regime model ($\lambda = 0$) and lower than those obtained in "bad" regime in one regime model ($\lambda = 0$). For the same λ, an assumption of starting in the "good" regime initially in the regime-switching model leads to lower spread rates compared to the assumption of starting in the "bad" regime initially, which reveals that a lower risk premium is required if the economy starts at the "good" state initially, and higher risk premiums are required to make up for higher default risk due to the possibility of switching to the "bad" regime in the regime-switching model.

Here it is evident that the parameter λ has a significant effect on the spread rates. If the economy starts at the "good" regime, the spread rates increase as λ increases, which may be caused by the fact that a higher risk premium is required to compensate for the higher risk of switching to the "bad" regime when λ increases. On the other hand, if we start at the "bad" regime, the spread rates decrease as λ increases. The reason may be that as λ increases it is more likely that the "bad" regime will switch to the "good" regime later. Because there is a lower probability of default in the "good" regime, the switching from the "bad" regime to the "good" regime results in a reduction of default risk, and thus results in a lower spread rate.

Figure 2. CDS spread rates for varying value of λ

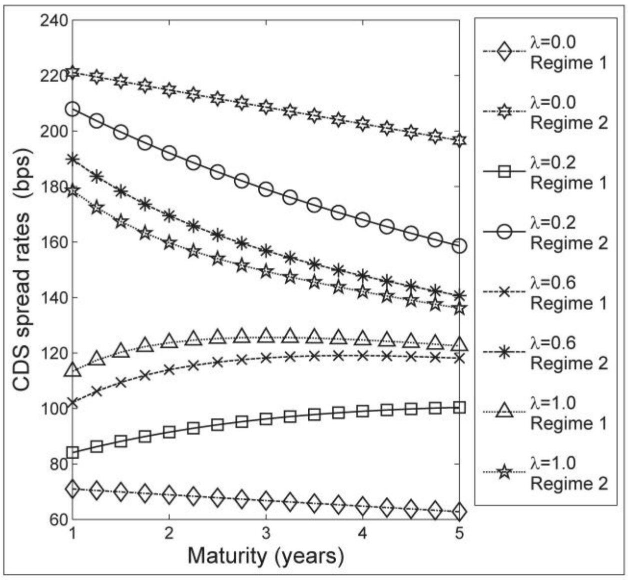
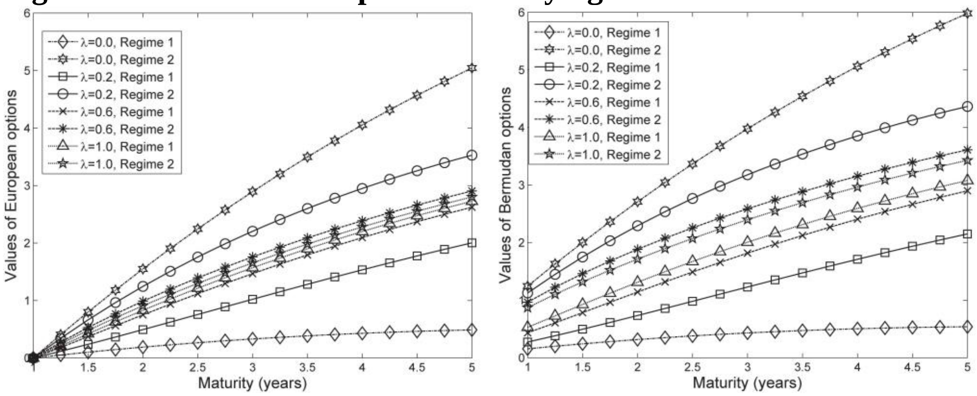


Figure 3 displays the impacts of regime-switching on the values of European and Bermudan CDS options with maturity $t_a = 1$ year for varying value of the parameter λ when the maturities of CDSs are 1, 1.25, 1.5, ⋯, 4.5, 4.75, 5 years. We can observe that the values of European and Bermudan CDS options in the regime-switching model are higher than those obtained in “good” regime in one regime model and lower than those obtained in “bad” regime in one regime model. For the same λ , the options’ values when the economy starts in the “bad” regime at time 0 are higher than those when the economy starts in the “good” regime at time 0. This may be attributed to a higher spread rate when we start at the “bad” regime as shown by Figure 2.

The parameter λ has a strike effect on the values of European and Bermudan CDS options. When the economy starts in the “good” regime, the options’ values increase as λ increases, for the spread rates increase with λ. On the other side, when we start at the “bad” regime, the options’ values decrease as λ increases, for the reason that the spread rates decrease with an increase of λ. For the following results we set the parameter in the generator matrix $\lambda = 0.6$.

Figure 3. Values of CDS options for varying value of λ

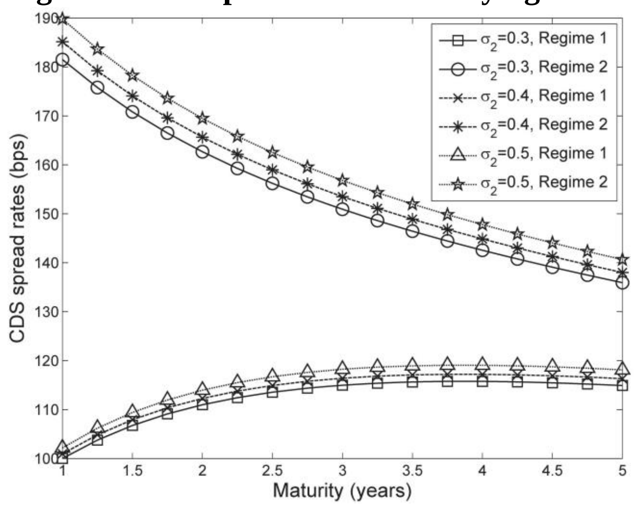


a. European options b. Bermudan options

Case 2: The impact of the volatility parameter σ_i

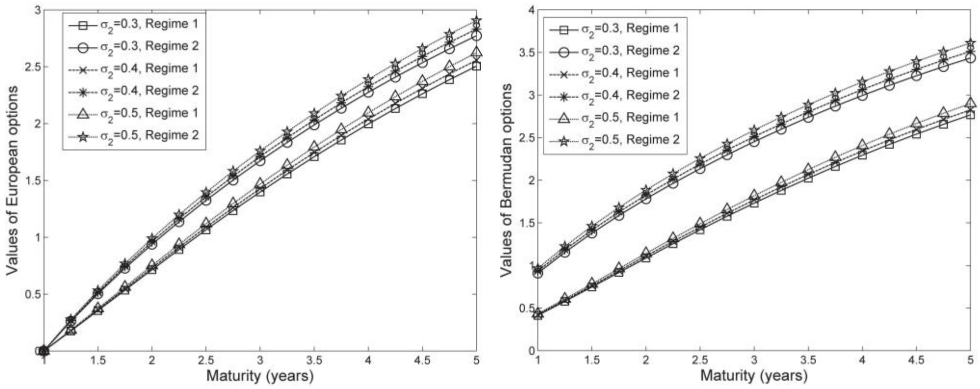
Figure 4 depicts the plots of the spread rates with different values of σ_2 in the regime-switching model. The spread rates increase as σ_2 increases when the economy starts in the "good" regime or starts in the "bad" regime, which reflects that as the volatility parameter σ_2 of the equity price becomes higher, the default probabilities will become higher, and thus a higher risk premium is required to compensate for the higher default risk.

Figure 4. CDS spread rates for varying value of σ_2



In Figure 5 a sensitivity analysis for the values of European and Bermudan CDS options as σ_2 varies from 0.3 to 0.5 is shown. It is observed that the values of European and Bermudan CDS options increase while σ_2 increases, when the economy starts in the "good" regime or starts in the "bad" regime, which may be ascribed to a higher spread rate with an increase of a_2 shown by Figure 4.

Figure 5. Values of European and Bermudan options for varying value of σ_2

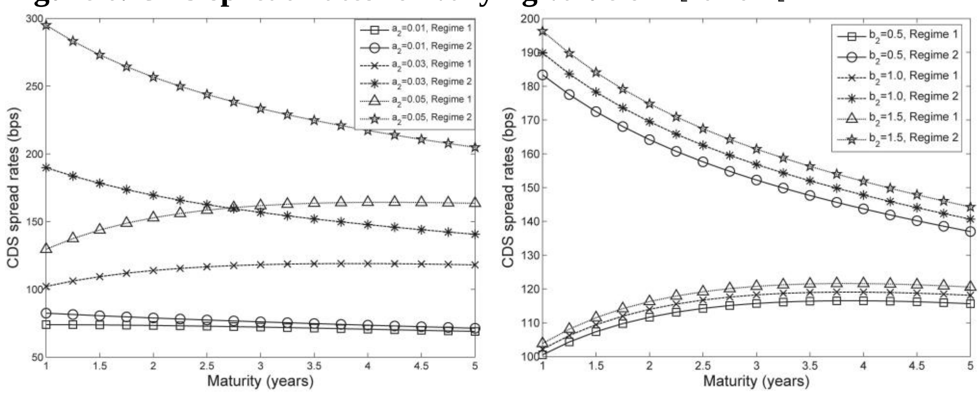


a. European options b.Bermudan options

Case 3: The impact of the default intensity parameters a_i and b_i

Figure 6 depicts the plots of the spread rates with different values of a_2 and b_2 while a_1, b_1 take constant values 0.01 and 0.5 in the regime-switching model. The spread rates increase as a_2 and b_2 increase when the economy starts in the "good" regime or starts in the "bad" regime, which reflects that, as a_2 and b_2 increase the default probabilities will become higher, and thus a higher risk premium is required to compensate for the higher default risk.

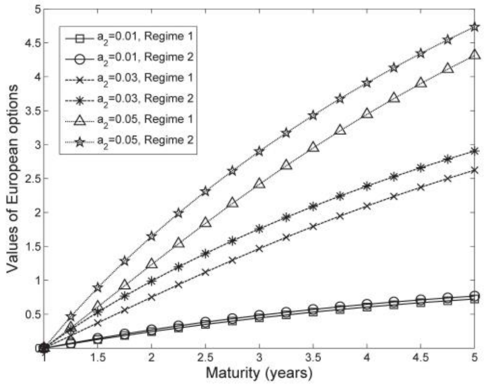
Figure 6. CDS spread rates for varying value of a_2 and b_2



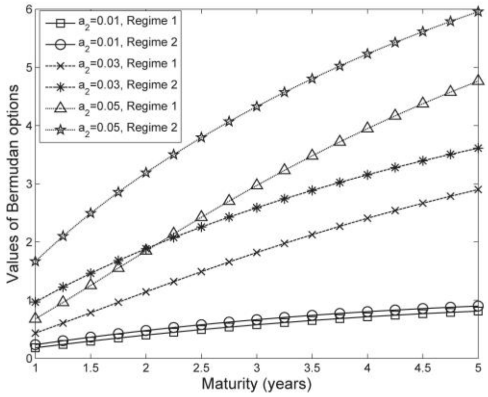
a. The impact of a_2 b. The impact of b_2

Figure 7 shows the values of European and Bermudan CDS options as a_2 varies from 0.3 to 0.5 and b_2 varies from 0.5 to 1.5 respectively, and a_1, b_1 take constant values 0.01 and 0.5. It is observed that the values of European and Bermudan CDS options increase while a_2 and b_2 increase, when the economy starts in the "good" regime or starts in the "bad" regime, which is due to a higher spread rate with an increase of a_2 and b_2 shown by Figure 6.

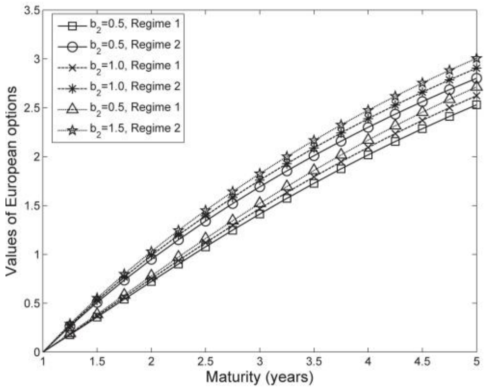
Figure 7. Values of European and Bermudan options for varying value of a_2 and b_2



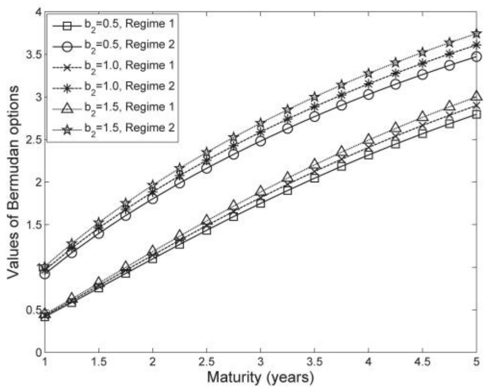
a. Impact of a_2 on European options



b. Impact of a_2 on Bermudan options



c. Impact of b_2 on European options



d. Impact of b_2 on Bermudan options

7. Conclusions

We studied the pricing of defaultable bonds, CDSs, and CDS options under a regime-switching CEV process with jump to default. Our findings based on the numerical experiment implied that: (1) the presence of a "bad" economic regime increases the spread rates of CDSs and the values of CDS options substantially; (2) the inclusion of the Markovian regime-switching effect is a possible way to explain and improve the underestimation of the empirical probability of default. We also performed various sensitivity analyses for the spread rates and CDS options when the elements of the generator matrix and other model parameters varied. We found that an increase in the value of λ leads to an increase in bonds' values, spread rates, and CDS options assuming we start at the "good" regime, and with a decreasing effect otherwise. The effects of model parameters on bonds' values, spread rates, and CDS options are significant, which should be taken into account when estimating the values of such products.

For further research it would be worth extending our model to include a regime-switching CEV process for modeling short rate dynamics or volatility dynamics. In this case how to develop lattice-based techniques to model the correlation of two processes and price credit derivatives such as equity default swaps and variance swaps is an interesting topic.

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