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A MODIFIED HARMONY SEARCH ALGORITHM FOR PORTFOLIO OPTIMIZATION PROBLEMS

Abstract*. For a diversified portfolio problem, building an optimization model is very necessary to make investment return be as large as possible and to make the investment risk be as small as possible. In this work, firstly, the basic mathematic model of Portfolio Optimization (PO) and Cardinality Constrained Mean–Variance (CCMV) model are introduced. Then a modified Harmony search algorithm called HSDS based on Dimensional-Selection (DS) strategy and dynamic fret width (FW) strategy is proposed to solve PO problems, in which the DS strategy is for avoiding generating invalid solutions and the FW strategy is to balance global exploration and local exploitation. Finally, Genetic Algorithm, Particle Swarm Optimization, Simulated Annealing and Tabu Search are compared with the HSDS algorithm employing five portfolio problems (HangSeng, DAX 100, FTSE 100, S&P 100 and Nikkei). Experimental results indicate that the proposed algorithm is very effective for solving large scale portfolio optimization problems.*

Keywords: *Portfolio Optimization; Harmony search Algorithm; Dimensional-selection strategy; Cardinality Constrained Mean-Variance Model;*

JEL Classification: G11

1. Introduction

Affected by financial crisis, the global economic is downturn. As prices rise year by year, traditional way of bank storage gets no profit, even causes the capital devaluation. In recent years, many people start to concentrate on varieties of investment and financing, such as common stocks or stock indices, domestic and foreign bonds or bond indices, foreign cash, real estate, commodities and so on. Any investment frequently has the double factors of returns and risk. So when investors select the target of investment, they must take the returns and risk into consideration. Consequently, how to select the most optimal portfolio can be a critical issue. For this reason, researchers put forward some portfolio optimization theory models. In [1], Markowitz presented a Mean-Variance (MV) Portfolio theory framework model, which analyzed the balance between the returns and risk systematically. Konno and Yamazaki [2] proposed a Mean-Absolute Deviation (MAD) model, whose advantage is that it can carry out the risk measure in any case. A Value at Risk (VaR) was put forward in literature [3] to control the market analysis. In literature [4], VaR model was improved, and a new model CVaR (Conditional Value at Risk) was introduced. A Cardinality Constrained Mean–Variance (CCMV) was proposed in literature [5].

However, these investment models are so complex even non-smooth optimal problems that can not be solved by traditional algorithm in gradient. To solve the portfolio optimization model, swarm intelligence optimization algorithms has been some successful applications. Genetic algorithm (GA) is applied to the optimal management of the investment funds in literature [7, 8]. J.D. Bermúdez [6] presented a multi-objective genetic optimization algorithm, which is applied to the constrained investment options. K.P. Anagnostopoulo [9] proposed a multi-objective evolutionary algorithm applied in solving the constraints on portfolio optimization problems. An investment optimization model based on genetic network programming is put forward in [10, 11]. Particle swarm optimization (PSO) algorithm [12-15, 17] and artificial bee colony (ABC) algorithm [16] are also applied to solve the portfolio optimization problems.

Harmony search (HS) Algorithm [18, 21] is a new swarm intelligence optimization algorithm inspired by the improvisation process of musicians proposed by Zong Woo Geem in 2001. In recent years, HS has obtained a wide range of applications. In this paper, a modified harmony search algorithm is proposed to solve the portfolio optimization problems.

The rest of this paper is organized in the following way: Section 2 introduces the basic mathematical model of portfolio optimization and CCMV model in detail. Section 3.1 reviews the standard harmony search algorithm in briefly. The proposed harmony search algorithm (HSDS) is presented to solve CCMV portfolio optimization problem in Section 3.2 and Section 3.3. Experimental results are investigated in Section 4. Finally, Section 5 summarizes this work.

2. Portfolio optimization model

Portfolios can be interpreted in a mean-variance (MV) framework, with every investor holding the portfolio with the lowest possible return variance (risk) consistent with that investor's chosen level of expected return (which is called a minimum-variance portfolio), if the returns on the assets are jointly elliptically distributed, including the special case in which they are jointly normally distributed[22][23]. Under MV analysis, it can be shown [24] that every minimum risk portfolio given a particular expected return can be formed as a combination of any two efficient portfolios. If the investor's optimal portfolio has an expected return that is between the expected returns on two efficient benchmark portfolios, then that investor's portfolio can be characterized as consisting of positive quantities of the two benchmark portfolios.

2.1 Mathematical model of portfolio optimization

Portfolio optimization is a process of choosing the proportions of various risky assets to be held in a portfolio according to some criterion, which attempts to maximize portfolio expected return for a given amount of portfolio risk, or equivalently minimize risk for a given level of expected return [19]. Assume that there are *D* assets available, we want to choose some of those to invest so as to obtain the expected return *R** with minimum risk. The standard mathematical model [20] of portfolio problem is described as follows:

$$\min f(X) = \sum_{i=1}^{D}\sum_{j=1}^{D} x_i x_j \delta_{ij} \tag{1}$$

s.t.

$$\sum_{i}^{D} x_i \mu_i = R^* \tag{2}$$

$$\sum_{i=1}^{D} x_i = 1, \quad 0 \le x_i \le 1, i = 1, 2, ..., D \tag{3}$$

where x_i is the portfolio weight which denotes the percentage of wealth invested in *i*-th asset and subjects to the budget constraint $\sum_{i=1}^{D} x_i = 1$, *D* represents the number of total assets available, μ_i denotes the expected return rate of the *i*-th asset (*i*=1,2,…,*D*), δ_{ij} is the covariance between the *i*-th asset and the *j*-th asset (i=1,2,…,*D*; j=1,2,…,*D*). The object function $f(X)$ is to obtain the optimal investment allocation $X^* = \left(x_1^*, x_2^*, \cdots, x_D^*\right)$ which can minimize the total risk associate with the expected return of *R**.

In the standard PO model, the desired number of investing-assets, the lower and the upper bounds of proportion of each asset in portfolio are not restricted, and the expected return is predetermined before the portfolio. In practice, the investors may have some constraint conditions, such as maximum return, desired number of investing-assets and the bounds of proportion of each asset.

2.2 CCMV portfolio optimization model

The portfolio optimization model with cardinal number constrained (CCMV) [5, 16] is based on Mean-Variance (MV) model, which takes the risk aversion parameter into consideration. The CCMV model is described as follows:

$$\min f(X) = (1-\lambda)\cdot \mathrm{R}^E - \lambda \cdot R^I \tag{4}$$

$$R^I = \left[\sum_{i=1}^{D}\sum_{j=1}^{D} x_i x_j \delta_{ij}\right] \tag{5}$$

$$R^E = \left[\sum_{i=1}^{D} x_i \mu_i\right] \tag{6}$$

S.t.

$$\sum_{i=1}^{D} z_i = K, \left(z_i \in \{0,1\}\right) \tag{7}$$

$$\sum_{i=1}^{D} x_i = 1, \left(0 \le x_i \le 1, i = 1, 2, ..., D\right) \tag{8}$$

$$\xi_i z_i \le x_i \le \zeta_i z_i, \left(\ i = 1, \cdots, D\right) \tag{9}$$

In Eq.(4), R^E denotes the total the return of investment, R^I represents the total risk in the investment, λ is the risk aversion parameter which is between 0 and 1.

The constraint conditions are described in Eqs.7-9.

In Eq.7, z_i denotes whether the ***i***-th asset can be chosen to invest, if z_i =1, the asset *i* is selected to invest, otherwise, it means that the asset *i* would be no included in the portfolio. *K* is the desired number of assets in the portfolio. The Eq.7 satisfies the constraint if and only if there are *K* assets be held exactly

The Eq.8 restricts that the sum of portfolio weight $x_i \left(i = 1, 2, \cdots, D\right)$ is equal to 1.

In Eq.9, the portfolio weight x_i is restricted within [ξ_i , ζ_i]. ξ_i and ζ_i are the minimum and the maximum proportion respectively, which ensures that x_i must lie between ξ_i and ζ_i if z_i =1, whilst $x_i = 0$ if z_i =0.

It can be seen from the Eq.4 that, when λ equals 0, the goal of optimization is to maximize the return of investment regardless of the investment risk. In contrast, when λ is equal to 1, the goal of model is to minimize risk of the portfolio regardless of the return. Surely, when we do the investment choice, we should take the return and the risk into consideration at the same time to make the return as large as possible and the risk as small as possible. Consequently, we should find a balance between the return and the risk. Each case with different value of λ has the corresponding expected return R^{E*} and the risk R^{I*}. So we can draw a continuous curve that is called an efficient frontier by tracing the return R^{E*} and the risk R^{I*} for varying values of λ .

The CCMV optimization model is a mixed quadratic and integer programming problem. With the increasing of total number *D* of assets, the calculation cost is very expensive. This work introduces a harmony search algorithm for solving portfolio optimization problem.

3. Harmony search algorithm for CCMV portfolio optimization model

3.1 Standard HS algorithm

Several important concepts in HS algorithm:

(1) Harmony memory (HM) is similar to the populations of the genetic algorithm, which randomly generated in the search space initially.

$$\text{HM}=\begin{bmatrix} X^1 \\ X^2 \\ \vdots \\ X^{\text{HMS}} \end{bmatrix}=\begin{bmatrix} x_1^1 & x_2^1 & \cdots & x_D^1 \\ x_1^2 & x_2^2 & \cdots & x_D^2 \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{\text{HMS}} & x_2^{\text{HMS}} & \cdots & x_D^{\text{HMS}} \end{bmatrix}$$

(2) Harmony memory consideration rate (HMCR): HMCR is the probability that the value of decision variable of new solution is selected from HM.

(3) Pitch-adjusting rate (PAR): PAR is the probability of adjusting a decision variable with fret width (*FW*).

Steps of standard HS algorithm are as follows:

(a) The method of improvising new harmony $X^{new}=(x_1^{new},x_2^{new},\cdots,x_D^{new})$ is following:

```
For i = 1 to D
    if rand<HMCR
    {   x_i^new = x_i^a; // a ∈ {1, 2, ···, HMS}
        if rand<PAR
        {  x_i^new = x_i^new + rand(−1,1) × FW(i);  }
    }
    else
    {   x_i^new = x_i^L + rand(0,1) × (x_i^U - x_i^L);   }
EndFor
```

(b) If the X^{new} is better than the worst harmony $X^{idworst}$ in harmony memory (*idworst* is the index of the worst harmony in HM), then replace $X^{idworst}$ with X^{new}.

(c) Checking the stopping criterion. If stopping criterion is satisfied, then computation is terminated. Otherwise, go to the step (a).

Standard HS algorithm has a strong global exploration capacity. But the precision of solution is unsatisfactory. For this, researchers have presented many variants of HS **[24]**.

3.2 Proposed algorithm

Due to high-dimensionality and complexity, it is very difficult for the portfolio optimization problems to obtain the optimal solution. Thus some swarm intelligent optimization algorithms [5-17] are employed to address this problem. However, most of algorithms are easily trapped into local search in solving PO problems.

In this paper, a modified HS algorithm based on dimensional-selection strategy is proposed to solve the portfolio optimization (PO) problems. The main idea of HSDS algorithm is that, in the beginning of search, more decision variables are adjusted with large probability of SP, which is conducive to maintain the global exploration ability; with the progress of search, the global exploration ability decreases gradually and the local exploitation capability increases step by step; In the later of searching, for obtaining the high-precision optimal solution, very few variables are adjusted with small probability of SP. The pseudo-code of HSDS algorithm is shown in Algortihm1,where *rand*(-1,1) denotes a uniformly distributed random number between -1 and 1; *rand*(0,1) is a uniformly distributed random number between 0 and 1; $rand(1,D)$ represents a *D*-dimensional vector that consist of *D* number of uniformly distributed random numbers between 0 and 1.

Algorithm1: HSDS algorithm

While t<T$_{\max}$

{ $X^{new} = X^{idworst}$;

$\mathrm{SP}=\mathrm{SP}_{\max}-(\mathrm{SP}_{\max}-\mathrm{SP}_{\min})\times\left(\frac{\mathrm{t}}{\mathrm{T}_{\max}}\right)^2$

S_D=**find** (*rand* (1, *D*) <SP).

For *i* in S_D

{ **If *rand* < HMCR**

{ $x_i^{new} = x_i^a, a \in \{1,2,\cdots,\mathrm{HMS}\}$;

If *rand* < PAR

{ $x_i^{new} = x_i^{new} + rand(-1,1)\cdot FW(i)$; **}**

Else

{ $x_i^{new} = x^L{}_i + rand(0,1)\times(x^U{}_i - x^L{}_i)$; **}**

}

t=t+1;

}

In the HSDS algorithm, parameters SP (see Eq.10), ***FW*** (see Eq.11) and PAR (see Eq.12) vary with the iteration *t* and they are calculated as follows.

$$\mathrm{SP(t)}=\mathrm{SP}_{\max}-(\mathrm{SP}_{\max}-\mathrm{SP}_{\min})\times\left(\frac{t}{\mathrm{T}_{\max}}\right)^2 \tag{10}$$

$$FW(t)=\begin{cases} FW_{\max}\times\left(\dfrac{FW_{\mathrm{mid}}}{FW_{\max}}\right)^{\left(\frac{t}{\mathrm{T}_{\max}/2}\right)}, & t\le\dfrac{\mathrm{T}_{\max}}{2} \\ FW_{\mathrm{mid}}\times\left(\dfrac{FW_{\min}}{FW_{\mathrm{mid}}}\right)^{\left(\frac{t-\mathrm{T}_{\max}/2}{\mathrm{T}_{\max}/2}\right)}, & t>\dfrac{\mathrm{T}_{\max}}{2} \end{cases} \tag{11}$$

$$\mathrm{PAR}(t)=\mathrm{PAR}_{\min}+(\mathrm{PAR}_{\max}-\mathrm{PAR}_{\min})\times\frac{t}{\mathrm{T}_{\max}} \tag{12}$$

The change curves of SP and ***FW*** are shown in Fig.1.

The ***FW*** and selection probability (SP) decreases gradually with the increasing of iterations. In the early stage of search, each decision variable of X^{new} has more opportunity to be chosen with a large value of SP, and the perturbation ability of space is strong with a large value of ***FW***, which is contribute to find new region in search space . as time *t* progresses, the selection probability(SP) of each decision variable and the value of ***FW*** decrease gradually, so the disturbance of space declines and the exploitation power of local search are enhanced increasingly.

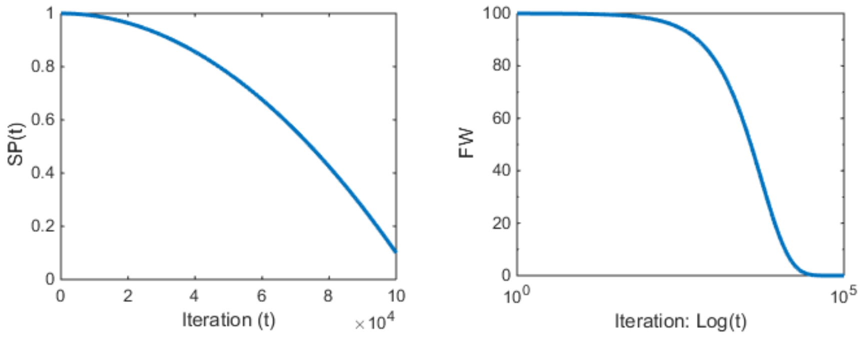


Figure1. The change curves of parameters SP and *FW*

3.3 Solving CCMV portfolio optimization problem with proposed algorithm

3.3.1 Efficient frontier 10

For the CCMV portfolio optimization problem, a harmony $X^i = \left(x_1^i, x_2^i, \cdots, x_D^i\right)$ represents an investment selection; the fitness function is expressed as Eq.4. It can be seen that each case with different value of λ has a optimal investment selection (R^{E*} , R^{I*}). The case λ =0 represents maximum expected return regardless of the risk involved, the optimal investment selection involves only the asset with the highest return; the case λ =1 is to minimize risk regardless of the investment return. Therefore, solving Eqs.4-9 for different value of λ $(0 \le \lambda \le 1)$, we trace out the corresponding optimal investment selection (R^{E*}, R^{I*}) which will construct an curve of efficient frontier . The curve gives the best possible tradeoff of risk against return; it denotes the set of the optimal portfolios.

3.3.2 Handling the constraint

CCMV Model is a constrained optimization problem. Be sure that the obtained optimal solution is practicable for solving it. Generally, the constraint processing technology is applied to the constraint optimization problem, such as penalty function method, but its computational cost is very large, and the result isn't ideal. It can be seen in Eqs.7-9 that the constraint condition is Mixed Integer Quadratic Program. For the constraint handling, we employ the method in literature [16-17].

If $K^* > K$, some assets that have been chosen is must be removed to satisfy Eq.(7). Two strategies are employed in this work, which include removing an asset randomly and selecting a minimum *c-valued* asset to remove.

Assume that Ψ is the set of all assets that can be chosen to invest, *S** is the set of assets which have been selected from Ψ. If $K^* = \sum_{i=1}^{D} z_i$ >*K*, then some assets must be removed from S*; if $K^* < K$, then some assets must be added into S* until $K^* = K$. The strategy which of the redundant assets is to be removed or which of the remaining assets is be added is as follows:

While $K^* > K$
if rand<0.5
select randomly an asset in S* and then remove it from S*
else
select the minimum *c*-valued asset in S* and then remove it from S*
end
While $K^* < K$
if rand<0.5
select an asset in Ψ-S* randomly and then add it into S*
else
select the maximum *c*-valued asset in Ψ-S* and then add it into S*
end

The formula of *c-value* (C_i) for asset *i* is shown as Eq.13:

$$C_i = \frac{u_i + \Omega}{\Delta_i + \Phi} \quad (\text{i}=1, 2, \ldots, D) \qquad (13)$$

where,

$$u_i = 1 + (1-\lambda)\mu_i$$

$$\Delta_i = 1 + \frac{\lambda \sum_{j=1}^{D} \delta_{ij}}{D}$$

$$\Omega = -\min(0, u_1, \cdots, u_D)$$

$$\Phi = -\min(0, \Delta_1, \cdots, \Delta_D)$$

The technique of boundary truncation is employed to handle constraint: $\xi_i z_i \le x_i \le \zeta_i z_i,\ z_i \in \{0,1\}, i = 1,\cdots,D$. For constraint $\sum_{i=1}^{D} x_i = 1$, the value of x_i is recalculated as follows.

$E = 0$, $E' = 0$;
for $i = 1 : D$
if $(z_i\zeta_i - x_i) > 0$
$E = E + (\zeta_i - x_i)$;
else
$E' = E' - (\zeta_i - x_i)$;
end
end

$\Gamma = 0$, $\Gamma' = 0$;
for $i = 1 : D$
if $(z_i\xi_i - x_i) < 0$
$\Gamma = \Gamma - (\xi_i - x_i)$;
else
$\Gamma' = \Gamma' + (\xi_i - x_i)$;
end
end

$$x_i = \begin{cases} x_i + (z_i\zeta_i - x_i)\dfrac{E'}{E}, & \text{if } z_i\zeta_i - x_i > 0 \\ \zeta_i, & \text{if } z_i\zeta_i - x_i < 0 \\ x_i - (z_i\xi_i - x_i)\dfrac{\Gamma'}{\Gamma}, & \text{if } z_i\xi_i - x_i < 0 \\ \xi_i, & \text{otherwise} \end{cases} \tag{14}$$

If $x_i > \zeta_i$ or $x_i < \xi_i$, then the Equation (14) will be performed recurrently until $\xi_i \le x_i \le \zeta_i$.

3.3.3 Solving CCMV portfolio optimization problem based on HSDS

The flow chart of solving portfolio optimization problem based on HSDS is shown in Fig.2.

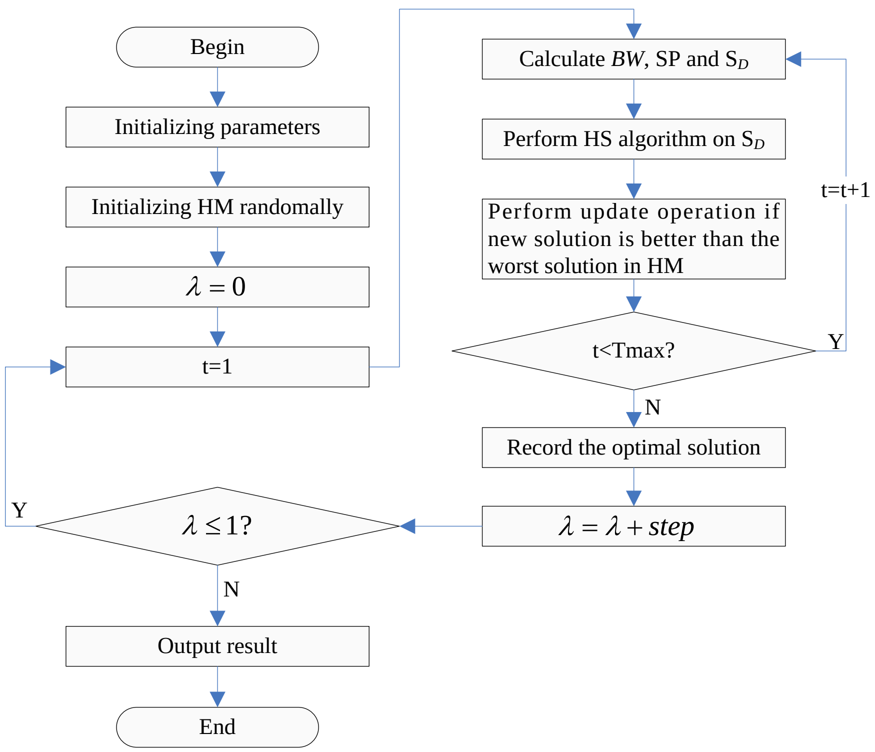


Figure 2. The flow chart of solving portfolio optimization problem based on HSDS

4. Experiments and discussion

4.1 Preparation of experiment

Five test data (HangSeng, DAX100, FTSE100, S&P100 and Nikkei) which are from http://people.brunel.ac.uk/~mastjjb/jeb/orlib/portinfo.html are employed to investigate the performance of the proposed algorithm. In the experiment, four typical swarm intelligent optimization algorithms [5] [16] [17] (PSO, GA, SA and TS) are compared with the HSDS algorithm. The experiment includes following two Tests.

(Test 1: Unconstraint portfolio optimization**)**. In Test 1, the desired number of assets in the portfolio is not constraint, and the investment proportion of each asset is also unconstraint; the value of λ varies from 0 to 1 with step 0.02.

(Test 2: constraint portfolio optimization**)**. In Test 2, the desired number of assets in the portfolio is 10, the lower limit ξ_i and the upper limit ζ_i of investment proportion of each asset are 0.01 and 1, respectively; the λ varies from 0 to 1 with step 0.02.

For Test 1 and Test 2, the parameters setting of HSDS are shown in **Table 1** ：

Table 1. The parameters setting of HSDS

HMS	HMCR	PAR	***FW***	SP	T_{max}
10	0.99	PAR_{max}=0.99 PAR_{min}=0.1	$\boldsymbol{FW}_{max}$=(xU-xL)/20 $\boldsymbol{FW}_{mid}$=(xU-xL)/(1e+4) $\boldsymbol{FW}_{min}$=(xU-xL)/(1e+15)	SP_{max}=0.6 SP_{min}=5/D	1000*D*

All the experiments were performed on Windows XP 32 system with Intel(R) Core(TM) i3-2120 CPU@3.30 GHz and 2 GB RAM, and all the program codes were written in MATLAB R2009a.

In this study, we trace out the sets of Pareto optimal portfolios obtained with each algorithm and compute the mean Euclidian distance from standard efficient frontier to the frontier of each algorithm, variance of returns error and mean return error. The mean Euclidian distance (*MED*), variance of returns error (*VRE*), and mean return error (*MRE*) are defined as follows [16, 17].

$$MED = \frac{1}{\xi}\sum_{j=1}^{\xi}\sqrt{\left(v_{i_j}^s - v_j^h\right)^2 + \left(r_{i_j}^s - r_j^h\right)^2}$$

$$VRE = \frac{1}{\xi}\left(\sum_{j=1}^{\xi}\frac{100\left|v_{i_j}^s - v_j^h\right|}{v_j^h}\right)$$

$$MRE = \frac{1}{\xi}\left(\sum_{j=1}^{\xi}\frac{100\left|r_{i_j}^s - r_j^h\right|}{r_j^h}\right)$$

where point $(v_i^s, r_i^s)(i = 1, 2, \cdots, 2000)$ denotes the variance and mean return in the standard efficient frontier, $(v_j^h, r_j^h)(j = 1, 2, \cdots, \xi)$ is the variance and mean return of point in the efficient frontiers of HSDS algorithm, $i_j = \underset{i=1,2,\cdots,2000}{\arg\min}\left(\sqrt{\left(v_i^s - v_j^h\right)^2 + \left(r_i^s - r_j^h\right)^2}\right), j = 1, 2, \cdots, \xi.$

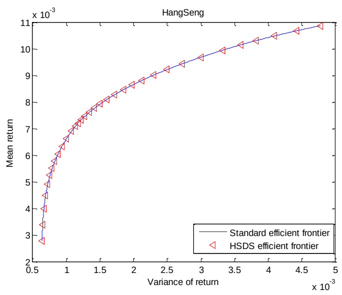
4.2 Experimental results and discussion

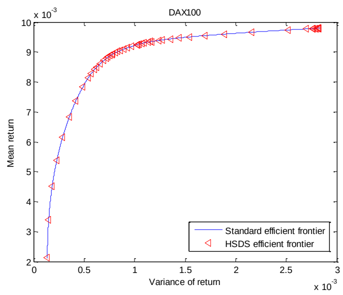
Table 2 shows the results of all algorithms in Test 1. the standard efficient frontiers and the efficient frontiers which are obtained by the HSDS algorithm are drawn in Figs.3.

For Test 2, the results of all algorithms are shown in Table 3, and Figs.4 draws the curve of the standard efficient frontiers and the curve of efficient frontiers obtained with HSDS algorithm.

Table 2. The experimental results of 5 algorithms in Test 1

Test data	Number of assets	Evaluation index	GA	PSO	TS	SA	HSDS
HangSeng	31	*MED*	5.90E-04	7.41E-04	5.98E-04	6.05E-04	9.71E-07
		VRE	2.90E-01	3.93E-01	2.90E-01	2.91E-01	2.51E-03
		MRE	1.06E-01	1.30E-01	1.07E-01	1.09E-01	1.01E-03
DAX100	85	*MED*	1.15E-03	1.36E-03	1.24E-03	1.18E-03	3.39E-06
		VRE	3.07E-01	3.93E-01	2.90E-01	2.91E-01	2.01E-01
		MRE	1.15E-01	1.30E-01	1.07E-01	1.09E-01	2.17E-02
FTSE100	89	*MED*	3.03E-04	3.33E-04	3.18E-04	3.25E-04	3.64E-06
		VRE	5.02E-01	5.36E-01	7.03E-01	6.69E-01	2.57E-01
		MRE	5.74E-02	6.38E-02	5.78E-02	5.79E-02	3.19E-02
S&P100	98	*MED*	6.20E-04	7.87E-04	6.20E-04	6.20E-04	3.86E-06
		VRE	6.10E-01	6.86E-01	1.00E+00	9.50E-01	2.88E-01
		MRE	2.13E-01	2.46E-01	1.25E-01	1.47E-01	2.68E-02
Nikkei	225	*MED*	1.50E-03	2.87E-04	1.51E-04	1.86E-04	1.01E-05
		VRE	2.11E-01	4.25E-01	2.18E-01	2.11E-01	1.84E-01
		MRE	9.33E-01	1.40E-01	7.37E-02	7.23E-02	5.90E-02





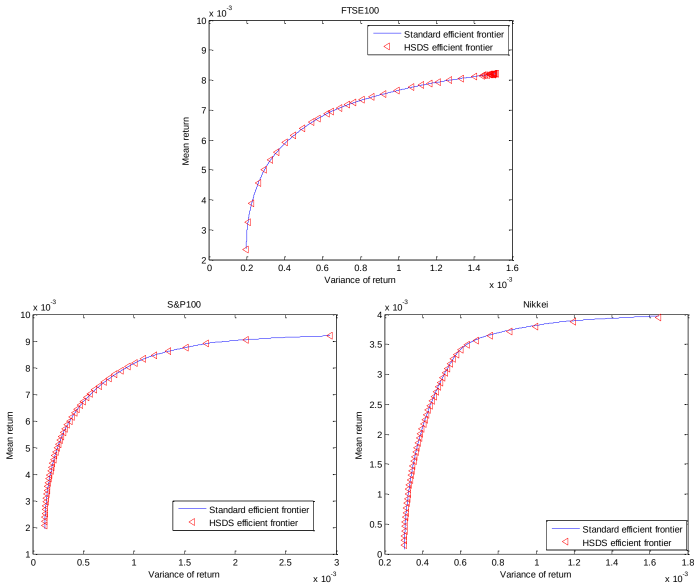


Figure 3. Efficient frontier comparison on Test 1

For Test 1, it can be seen clearly from Table 2 that, the mean Euclidian distance, variance of returns error and mean return error of HSDS algorithm are the minimum value of five algorithms. In Fig.3, the efficient frontier of HSDS is almost entirely overlapped with the standard efficient frontier.

Table 3. The experimental results of five algorithms in Test 2

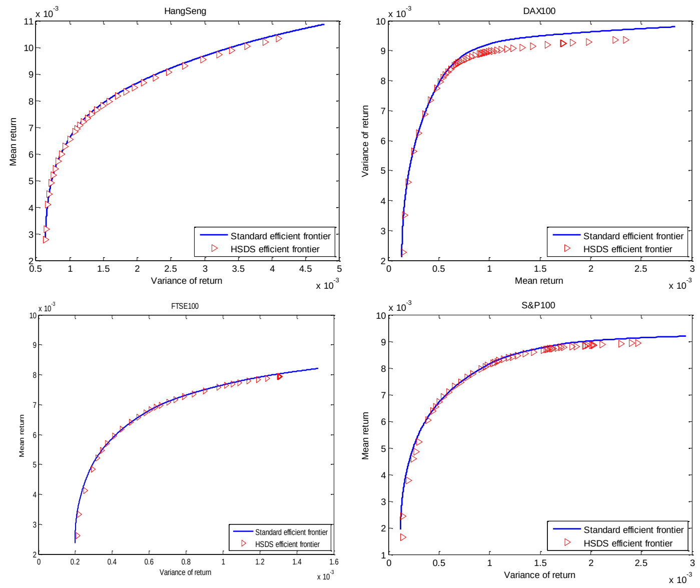
Test data	Num of assets	Evaluation index	GA	PSO	TS	SA	HSDS
Hang-Seng	31	*MED*	3.90E-03	4.90E-03	3.95E-03	4.00E-03	7.73E-05
		VRE	1.65E+00	2.24E+00	1.66E+00	1.66E+00	1.62E+00
		MRE	6.07E-01	7.43E-01	6.11E-01	6.24E-01	6.05E-01
DAX100	85	*MED*	7.60E-03	9.00E-03	8.20E-03	7.80E-03	1.47E-04
		VRE	1.75E+00	2.24E+00	1.66E+00	1.66E+00	1.26E+00
		MRE	6.57E-01	7.43E-01	6.11E-01	6.24E-01	7.09E-01
FTSE100	89	*MED*	2.00E-03	2.20E-03	2.10E-03	2.15E-03	3.72E-05
		VRE	2.87E+00	3.06E+00	4.01E+00	3.82E+00	2.66E+00
		MRE	3.28E-01	3.64E-01	3.30E-01	3.30E-01	3.94E-01
S&P100	98	*MED*	4.10E-03	5.20E-03	4.10E-03	4.10E-03	7.34E-05
		VRE	3.48E+00	3.91E+00	5.71E+00	5.42E+00	3.60E+00
		MRE	1.22E+00	1.40E+00	7.13E-01	8.42E-01	9.75E-01

		MED	9.93E-03	1.90E-03	1.00E-03	1.23E-03	7.73E-05
Nikkei	225	*VRE*	1.21E+00	2.43E+00	1.24E+00	1.20E+00	1.18E+00
		MRE	5.33E+00	8.00E-01	4.21E-01	4.13E-01	6.05E-01

For Test 2, it can be found from Table 3 that the *MED* of HSDS algorithm is the smallest one in five algorithms (GA, SA, TS, PSO and HSDS); and most of the *VRE* and *MRE* of HSDS algorithm are also the minimum value of five compared algorithms. In Fig.4, the efficient frontier of HSDS is very close to the standard efficient frontier (the Standard efficient frontier is the ideal result that the constraint didn't be yet considered).

5. Conclusion

In this paper, the study is focused on solving the large scale portfolio optimization problem. The standard mathematic model of portfolio optimization and Cardinality constrained Mean–Variance (CCMV) models are analyzed in detail. The CCMV is used to develop harmony search algorithm based on dynamic dimensional-selection strategy for portfolio optimization problem. The unconstraint and constraint experimental results are respectively compared to those obtained from intelligent optimization methods based on GA, PSO, SA and TS. The experimental results show that, the HSDS algorithm can obtain better solutions than the other methods. It indicates the HSDS algorithm is competitive method for solving large scale portfolio optimization problems.



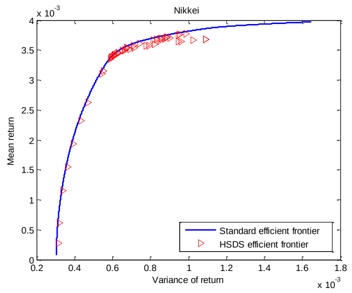


Figure 4. Efficient frontier comparison on Test 2

ACKNOWLEDGMENTS

The author would like to thank all the editors and referees for their valuable comments. This work was supported by the National Natural Science Foundation of China under Grant Nos.11401357, and the Scientific Research Program funded by the Projects Program of Shaanxi University of Technology Academician Workstation (No. fckt201509).

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