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MULTIPLE ATTRIBUTE GROUP DECISION MAKING METHODS BASED ON INTUITIONISTIC FUZZY GENERALIZED HAMACHER AGGREGATION OPERATOR

Abstract. *With respect to multiple attribute group decision making (MAGDM) problems in which attribute values take the form of the intuitionistic fuzzy values(IFVs), the group decision making method based on some generalized Hamacher aggregation operators which generalized the arithmetic aggregation operators and geometric aggregation operators and extended the Algebraic aggregation operators and Einstein aggregation operators, is developed. Firstly, the generalized intuitionistic fuzzy Hamacher weighted averaging(IFGHWA) operator, intuitionistic fuzzy generalized Hamacher ordered weighted averaging(IFGHOWA) operator, and intuitionistic fuzzy generalized Hamacher hybrid weighted averaging(IFGHHWA) operator, were proposed, and some desirable properties of these operators, such as commutativity, idempotency, monotonicity and boundedness, were studied. At the same time, some special cases in these operators were analyzed. Furthermore, one method to multi-criteria group decision-making based on these operators was developed, and the operational processes were illustrated in detail. Finally, an illustrative example is given to verify the proposed methods and to demonstrate their practicality and effectiveness.*

Keywords: *group decision-making, intuitionistic fuzzy values, Hamacher aggregation operators.*

JEL Classification: C44, C60

1. Introduction

There are a large number of multiple attribute group decision making (MAGDM) problems in real world, and these decision making problems are usually fuzzy and uncertain, their attribute values are more suitable to be expressed by fuzzy numbers. Since Zadeh (1965) firstly proposed fuzzy set theory, it has been a rapid development. Based on fuzzy set theory, Atanassov (1986, 1989) proposed the intuitionistic fuzzy set (IFS) which is a generalization of the fuzzy set. An important characteristic of IFS is that it is composed by a membership function and a non-membership function. Since its appearance, it has received more and more attentions, and many research results have been achieved. Atanassov (1994) defined some basic operations and relations of IFSs; Chen and Tan (1994) defined the score function of intuitionistic fuzzy value (IFV) so as to compare two IFVs, and Hong and Choi (2000) found that the score function alone cannot differentiate many IFVs, then they defined the accuracy function. Chen (2012) proposed several optimistic and pessimistic point operators, and presented an approach that relates optimism and pessimism to multi-criteria decision analysis in an intuitionistic fuzzy-decision environment.

The information aggregation operators are an interesting and important research topic, which are receiving more and more attentions (Liu 2011; Liu and Jin 2012a, 2012b; Liu and Su 2010; Xu 2007; Xu and Xia 2011; Xu and Yager 2006; Zhao et al. 2010). Xu (2007) proposed some arithmetic aggregation operators for intuitionistic fuzzy information. Then based on these operators, they presented the multiple attribute group decision making method. Xu and Yager (2006) proposed the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and the intuitionistic fuzzy hybrid geometric (IFHG) operator. Zhao (2010) proposed some new generalized aggregation operators for intuitionistic fuzzy information which can generalize arithmetic aggregation operators and geometric aggregation operators. Xu and Xia (2011) applied Choquet integral and Dempster-Shafer theory of evidence to aggregate inuitionistic fuzzy information, and proposed the induced generalized intuitionistic fuzzy Choquet integral operators and induced generalized intuitionistic fuzzy Dempster-Shafer operators. All above aggregation operators are based on the algebraic operational rules of intuitionistic fuzzy numbers(IFNs), and the key of the algebraic operations are Algebraic product and Algebraic sum, which are one type of operations that can be chosen to model the intersection and union of IFNs. In general, a t-norm and t-conorm can be used to model the intersection and union of IFNs (.Xia et al. 2012). Wang and Liu (2011) proposed the intuitionistic fuzzy aggregation operators based on Einstein operations which meet the typical t-norm and t-conorm and have the same smooth approximations as the algebraic operators. Hamacher t-conorm and t-norm are the generalization of algebraic and Einstein t-conorm and t-norm (Beliakov et al. 2007) and the generalized aggregation operators are the generalization of the arithmetic aggregation operators and geometric aggregation operators. Liu (2014) proposed Some Hamacher aggregation operators for the interval-valued intuitionistic fuzzy information. Liu et al. (2014) proposed some Hamacher aggregation operators for neutrosophic numbers and applied them to MAGDM problems.

Because intuitionistic fuzzy sets are the simple and effective way to express fuzzy information, and the generalized aggregation operators and

Hamacher operations are the more generalized than the existing operators. So combining generalized aggregation operators and Hamacher operations, we will develop some generalized Hamacher aggregation operators based on intuitionistic fuzzy information, which are the generalizations of the existing intuitionistic fuzzy aggregation operators.

2. Preliminaries

2.1. Intuitionistic fuzzy set

Definition 1(Atanassov1986**)**. Let $X=\{x_1,x_2,\cdots,x_n\}$ be a universe of discourse, an intuitionistic fuzzy set (IFS) A in X is given by

$$A=\{<x,u_A(x),v_A(x)>x\in X\} \tag{1}$$

where $u_A:X\rightarrow[0,1]$ and $v_A:X\rightarrow[0,1]$, with the condition $0\le u_A(x)+v_A(x)\le 1$, $\forall x\in X$ .The numbers $u_A(x)$ and $v_A(x)$ are the membership degree and non-membership degree of the element x to the set A, respectively.

Given an element x of X, the pair $(u_A(x),v_A(x))$ is called an intuitionistic fuzzy value (IFV) (Xu and Xia 2011). For convenience, it can be denoted as $\tilde{a}=(u_{\tilde{a}},v_{\tilde{a}})$ such that $u_{\tilde{a}}\in[0,1]$, $v_{\tilde{a}}\in[0,1]$ and $0\le u_{\tilde{a}}+v_{\tilde{a}}\le 1$.

Definition 2(Chen and Tan 1994**).**Let $\tilde{a}=(u_{\tilde{a}},v_{\tilde{a}})$ be an IFV, a score function S of the IFV $\tilde{a}$ is defined as the difference of membership and non-membership function, as follows:

$$S(\tilde{a})=u_{\tilde{a}}-v_{\tilde{a}} \tag{2}$$

where $S(\tilde{a})\in[-1,1]$.

Definition 3(Hong and Choi 2000**).**Let $\tilde{a}=(u_{\tilde{a}},v_{\tilde{a}})$ be an IFV, an accuracy function H of the IFV $\tilde{a}$ is defined as follows:

$$H(\tilde{a})=u_{\tilde{a}}+v_{\tilde{a}} \tag{3}$$

where $H(\tilde{a})\in[0,1]$.

Definition 4 (Xu 2007).If $\tilde{a}_1=(u_1,v_1)$ and $\tilde{a}_2=(u_2,v_2)$ are any two IFVs, $S(\tilde{a}_1)=u_1-v_1$ and $S(\tilde{a}_2)=u_2-v_2$ are the scores of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, and $H(\tilde{a}_1)=u_1+v_1$, $H(\tilde{a}_2)=u_2+v_2$ are the accuracy degrees of $\tilde{a}_1$ and $\tilde{a}_2$, respectively. Then,

(1) If $S(\tilde{a}_1)>S(\tilde{a}_2)$, then, $\tilde{a}_1>\tilde{a}_2$

(2) If $S(\tilde{a}_1)=S(\tilde{a}_2)$, then

If $H(\tilde{a}_1)>H(\tilde{a}_2)$, then $\tilde{a}_1>\tilde{a}_2$;

If $H(\tilde{a}_1)=H(\tilde{a}_2)$, then $\tilde{a}_1=\tilde{a}_2$.

2.2. Hamacher operators

The t-operators are in fact Intersection and Union operators in fuzzy set theory which are symbolized by T-norm (T), and T-conorm (T^*), respectively (Roychowdhury and Wang 1998).Based on a T-norm and T-conorm, a generalized union and a generalized intersection of intuitionistic fuzzy sets were introduced by Deschrijver and Kerre (2002).

Definition5 (Deschrijver and Kerre 2002)**.**Let A and B are any two intuitionistic fuzzy sets, then, the generalized intersection and union are defined as follows:

$$A\bigcap_{T,T^*} B=\{<x,T(u_A(x),u_B(x)),T^*(v_A(x),v_B(x))>x\in X\} \tag{4}$$

$$A\bigcup_{T,T^*} B=\{<x,T^*(u_A(x),u_B(x)),T(v_A(x),v_B(x))>x\in X\} \tag{5}$$

where T denotes a T-norm and T^* a T-conorm.

Some application examples of T-norms and T-conorms are listed as follows (Wang and Liu 2011):

(1) Algebraic T-norm and T-conorm

$$T(x,y)=x\times y \text{ and } T^*(x,y)=x+y-x\times y \tag{6}$$

As an application of Algebraic T-norm and T-conorm, suppose $\tilde{a}_1=(a_1,b_1)$ and $\tilde{a}_2=(a_2,b_2)$ are two IFVs, the algebraic product $\tilde{a}_1\otimes_A\tilde{a}_2$ and the algebraic sum $\tilde{a}_1\oplus_A\tilde{a}_2$ on two IFVs $\tilde{a}_1$ and $\tilde{a}_2$ can be obtained by defining T-norm and T-conorm. When $T(x,y)=xy$ and $T^*(x,y)=x+y-xy$, we can get

$$\tilde{a}_1\oplus\tilde{a}_2=(a_1+a_2-a_1a_2,b_1b_2) \tag{7}$$

$$\tilde{a}_1\otimes\tilde{a}_2=(a_1a_2,b_1+b_2-b_1b_2) \tag{8}$$

$$n\tilde{a}_1=\left(1-(1-a_1)^n,b_1^n\right)\quad n>0 \tag{9}$$

$$\tilde{a}_1^n=\left(a_1^n,1-(1-b_1)^n\right)\quad n>0 \tag{10}$$

Obviously, above operational laws are the same as those given by Atanassov (1986, 1989).

(2) Einstein T-norm and T-conorm

$$T(x,y)=\frac{x\times y}{1+(1-x)\times(1-y)}\text{, and } T^*(x,y)=\frac{x+y}{1+x\times y} \tag{11}$$

Further, Hamacher proposed a more generalized T-norm and T-conorm and they are defined as follows (Hamachar 1978).

$$T_\gamma(x,y)=\frac{xy}{\gamma+(1-\gamma)(x+y-xy)},\gamma>0 \tag{12}$$

$$T_\gamma^*(x,y)=\frac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy},\gamma>0 \tag{13}$$

Especially, when $\gamma = 1$, then Hamacher T-norm and T-conorm will reduce to $T(x,y) = xy$ and $T^*(x,y) = x + y - xy$ which are the Algebraic T-norm and T-conorm respectively; when $\gamma = 2$, then Hamacher T-norm and T-conorm will reduce to $T(x,y) = \frac{xy}{1+(1-x)(1-y)}$, and $T^*(x,y) = \frac{x+y}{1+xy}$ which are the Einstein T-norm and T-conorm respectively (Wang and Liu 2011).

3. Hamacher operations of IFVs

3.1. The operational rules based on Hamacher T-norm and T-conorm

Based on the Definition 5, Hamacher T-norm and T-conorm, we can establish the Hamacher product and Hamacher sum of two IFVs, respectively.

Let $\tilde{a}_1 = (a_1, b_1)$ and $\tilde{a}_2 = (a_2, b_2)$ be two IVIFNs, and $\gamma > 0$, then the operational rules based on Hamacher T-norm and T-conorm are defined as follows.

$$\tilde{a}_1 \oplus_h \tilde{a}_2 = \left(\frac{a_1 + a_2 - a_1 a_2 - (1-\gamma) a_1 a_2}{1-(1-\gamma)a_1 a_2}, \frac{b_1 b_2}{\gamma + (1-\gamma)(b_1 + b_2 - b_1 b_2)} \right) \quad (14)$$

$$\tilde{a}_1 \otimes_h \tilde{a}_2 = \left(\frac{a_1 a_2}{\gamma + (1-\gamma)(a_1 + a_2 - a_1 a_2)}, \frac{b_1 + b_2 - b_1 b_2 - (1-\gamma) b_1 b_2}{1-(1-\gamma) b_1 b_2} \right) \quad (15)$$

$$n\tilde{a}_1 = \left(\frac{\left(1+(\gamma-1)a_1\right)^n - (1-a_1)^n}{\left(1+(\gamma-1)a_1\right)^n + (\gamma-1)(1-a_1)^n}, \frac{\gamma b_1^n}{\left(1+(\gamma-1)(1-b_1)\right)^n + (\gamma-1)b_1^n} \right) \quad n > 0 \quad (16)$$

$$\tilde{a}_1^n = \left(\frac{\gamma a_1^n}{\left(1+(\gamma-1)(1-a_1)\right)^n + (\gamma-1)a_1^n}, \frac{\left(1+(\gamma-1)b_1\right)^n - (1-b_1)^n}{\left(1+(\gamma-1)b_1\right)^n + (\gamma-1)(1-b_1)^n} \right) \quad n > 0 \quad (17)$$

Theorem 1. Let $\tilde{a}_1 = (a_1, b_1)$ and $\tilde{a}_2 = (a_2, b_2)$ be any two IFVs, and $\gamma > 0$, then:

(1) $\tilde{a}_1 \oplus_h \tilde{a}_2 = \tilde{a}_2 \oplus_h \tilde{a}_1$ (18)

(2) $\tilde{a}_1 \otimes_h \tilde{a}_2 = \tilde{a}_2 \otimes_h \tilde{a}_1$ (19)

(3) $\eta(\tilde{a}_1 \oplus_h \tilde{a}_2) = \eta\tilde{a}_1 \oplus_h \eta\tilde{a}_2, \eta \geq 0$ (20)

(4) $\eta_1\tilde{a}_1 \oplus_h \eta_2\tilde{a}_1 = (\eta_1 + \eta_2)\tilde{a}_1, \ \eta_1, \eta_2 \geq 0$ (21)

(5) $\tilde{a}_1^{\eta_1} \otimes_h \tilde{a}_1^{\eta_2} = (\tilde{a}_1)^{\eta_1 + \eta_2}, \ \eta_1, \eta_2 \geq 0$ (22)

(6) $\tilde{a}_1^{\eta} \otimes_h \tilde{a}_2^{\eta} = (\tilde{a}_1 \otimes_h \tilde{a}_2)^{\eta}, \ \eta \geq 0$ (23)

It is easy to prove the formulas in Theorem 1, omitted in here.

3.2. The intuitionistic fuzzy generalized Hamacher hybrid averaging operators

We can give the definition of the intuitionistic fuzzygeneralized Hamacher averaging operators.

Definition 6. Let $\tilde{a}_j = \left(a_j, b_j\right)$ $(j = 1, 2 \cdots, n)$ be a collection of the IFVs, and $IFGHWA: \Omega^n \to \Omega$, if

$$IFGHWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\bigoplus_{j=1}^{n}{}_h \left(w_j \tilde{a}_j^{\lambda} \right) \right)^{1/\lambda} \tag{24}$$

where Ω is the set of all IFVs, and $\lambda > 0$. $w = \left(w_1, w_2, \cdots, w_n\right)^T$ is weight vector of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$, such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Then $IFGHWA$ is called the intuitionistic fuzzy generalized Hamacher weighted averaging operator.

Based on the Hamacher operational rules of the IFVs, we can derive the result shown as theorem 2.

Theorem 2. Let $\tilde{a}_j = \left(a_j, b_j\right)$ $(j = 1, 2 \cdots, n)$ be a collection of the IFVs and $\lambda > 0$,then the result aggregated from Definition 6 is still an IFV, and even

$$IFGHWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$

$$= \left(\frac{\gamma \left(\prod_{j=1}^{n} x_j^{w_j} - \prod_{j=1}^{n} y_j^{w_j} \right)^{1/\lambda}}{\left(\prod_{j=1}^{n} x_j^{w_j} + (\gamma^2 - 1) \prod_{j=1}^{n} y_j^{w_j} \right)^{1/\lambda} + (\gamma - 1) \left(\prod_{j=1}^{n} x_j^{w_j} - \prod_{j=1}^{n} y_j^{w_j} \right)^{1/\lambda}}, \right. \tag{25}$$

$$\left. \frac{\left(\prod_{j=1}^{n} z_j^{w_j} + (\gamma^2 - 1) \prod_{j=1}^{n} t_j^{w_j} \right)^{1/\lambda} - \left(\prod_{j=1}^{n} z_j^{w_j} - \prod_{j=1}^{n} t_j^{w_j} \right)^{1/\lambda}}{\left(\prod_{j=1}^{n} z_j^{w_j} + (\gamma^2 - 1) \prod_{j=1}^{n} t_j^{w_j} \right)^{1/\lambda} + (\gamma - 1) \left(\prod_{j=1}^{n} z_j^{w_j} - \prod_{j=1}^{n} t_j^{w_j} \right)^{1/\lambda}} \right)$$

where $x_j = \left(1 + (\gamma - 1)(1 - a_j)\right)^{\lambda} + (\gamma^2 - 1) a_j^{\lambda}$, $y_j = \left(1 + (\gamma - 1)(1 - a_j)\right)^{\lambda} - a_j^{\lambda}$ $z_j = \left(1 + (\gamma - 1) b_j\right)^{\lambda} + (\gamma^2 - 1)(1 - b_j)^{\lambda}$, $t_j = \left(1 + (\gamma - 1) b_j\right)^{\lambda} - (1 - b_j)^{\lambda}$, here $\gamma > 0$.

This Theorem can be proved by Mathematical induction, it is omitted here.

It is easy to prove that the $IFGHWA$ operator has the properties, such asmonotonicity, idempotency and boundedness.

Now we can discuss some special cases of the $IFGHWA$ operator with respect to the parameters λ and γ.

(1) If $\lambda = 1$, then the IFGHWA operator (24) will be reduced to the Hammer intuitionistic fuzzy weighted averaging (HIFWA) operator which is defined by Xia et al. (2012). According to (25), we can get

$$HIFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$
$$= \left(\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)a_j\right)^{w_j} - \prod_{j=1}^{n}(1-a_j)^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)a_j\right)^{w_j} + (\gamma-1)\prod_{j=1}^{n}(1-a_j)^{w_j}}, \frac{\gamma\prod_{j=1}^{n} b_j^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(1-b_j)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{n} b_j^{w_j}} \right) \quad (26)$$

Further,

(i) When $\gamma = 1$, the formula (26) will be reduced to an intuitionistic fuzzy weighted averaging (IFWA) operator which is defined by Xu (2007). It is shown as follows:

$$IFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(1 - \prod_{j=1}^{n}(1-a_j)^{w_j}, \prod_{j=1}^{n} b_j^{w_j} \right).$$

(ii) When $\gamma = 2$, the formula (26) will be reduced to the Einstein intuitionistic fuzzy weighted averaging (EIFWA) operator which is defined by Xia et al. (2012). It is shown as follows:

$$EIFWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\frac{\prod_{j=1}^{n}\left(1+a_j\right)^{w_j} - \prod_{j=1}^{n}(1-a_j)^{w_j}}{\prod_{j=1}^{n}\left(1+a_j\right)^{w_j} + \prod_{j=1}^{n}(1-a_j)^{w_j}}, \frac{2\prod_{j=1}^{n} b_j^{w_j}}{\prod_{j=1}^{n}\left(2-b_j\right)^{w_j} + \prod_{j=1}^{n} b_j^{w_j}} \right)$$

(2) If $\lambda \to 0$, then the IFGHWA operator (24) will be reduced to the Hammer intuitionistic fuzzy weighted geometric (HIFWG) operator which is defined by Xia et al. (2012). According to (25), we can get

$$HIFWG(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$
$$= \left(\frac{\gamma\prod_{j=1}^{n} a_j^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(1-a_j)\right)^{w_j} + (\gamma-1)\prod_{j=1}^{n} a_j^{w_j}}, \frac{\prod_{j=1}^{n}\left(1+(\gamma-1)b_j\right)^{w_j} - \prod_{j=1}^{n}(1-b_j)^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)b_j\right)^{w_j} + (\gamma-1)\prod_{j=1}^{n}(1-b_j)^{w_j}} \right) \quad (27)$$

Further,

(i) When $\gamma = 1$, the formula (27) will be reduced to an intuitionistic fuzzy weighted geometric (IFWG) operator which is defined by Xu and Yager (2006). It is shown as follows:

$$IFWG(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\prod_{j=1}^{n}a_j^{w_j},1-\prod_{j=1}^{n}(1-b_j)^{w_j}\right).$$

(ii) When $\gamma=2$, the formula (27) will be reduced to the Einstein intuitionistic fuzzy weighted geometric (EIFWG) operator which is defined by Xia et al. (2012). It is shown as follows:

$$EIFWG(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\frac{2\prod_{j=1}^{n}a_j^{w_j}}{\prod_{j=1}^{n}\left(2-a_j\right)^{w_j}+\prod_{j=1}^{n}a_j^{w_j}},\frac{\prod_{j=1}^{n}\left(1+b_j\right)^{w_j}-\prod_{j=1}^{n}(1-b_j)^{w_j}}{\prod_{j=1}^{n}\left(1+b_j\right)^{w_j}+\prod_{j=1}^{n}(1-b_j)^{w_j}}\right)$$

(3) If $\gamma=1$, then the IFGHWA operator (24) will be reduced to the generalized intuitionistic fuzzy weighted averaging (GIFWA) operator which is defined by Zhao et al. (2010). According to (25), we can get

$$GIFWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$$
$$=\left(\left(1-\prod_{j=1}^{n}\left(1-a_j^{\lambda}\right)^{w_j}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-b_j\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}\right) \tag{28}$$

(4) If $\gamma=2$, then the IFGHWA operator (24) will be reduced to the generalized Einstein intuitionistic fuzzy weighted averaging (GEIFWA) operator. According to (25), we can get

$$IFGHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$$
$$=\left(\frac{2\left(\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}x_j^{w_j}+3\prod_{j=1}^{n}y_j^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}x_j^{w_j}-\prod_{j=1}^{n}y_j^{w_j}\right)^{1/\lambda}},\frac{\left(\prod_{j=1}^{n}z_j^{w_j}+3\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}z_j^{w_j}-\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}z_j^{w_j}+3\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{n}z_j^{w_j}-\prod_{j=1}^{n}t_j^{w_j}\right)^{1/\lambda}}\right) \tag{29}$$

Where $x_j=\left(2-a_j\right)^{\lambda}+3a_j^{\lambda}$, $y_j=\left(2-a_j\right)^{\lambda}-a_j^{\lambda}$, $z_j=\left(1+b_j\right)^{\lambda}+3(1-b_j)^{\lambda}$,

$t_j=\left(1+b_j\right)^{\lambda}-(1-b_j)^{\lambda}$.

From the above description, we can know $IFGHWA$ operator is more generalized.

Definition 7. Let $\tilde{a}_j=\left(a_j,b_j\right)$ $(j=1,2\cdots,n)$ be a collection of the IFVs, and $IFGHOWA:\Omega^n\to\Omega$, if

$$IFGHOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(\bigoplus_{h\, j=1}^{n} \left(\omega_j \tilde{a}_{\sigma(j)}^{\lambda} \right) \right)^{1/\lambda} \tag{30}$$

where Ω is the set of all IFVs, and $\lambda > 0$. $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weighted vector associated with $IFGHOWA$, such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for any j Then $IFGHOWA$ is called the intuitionistic fuzzy generalized Hamacher ordered weighted averaging (IFGHOWA) operator.

Based on the Hamacher operational rules of the IFVs, we can derive the result shown as theorem 3.

Theorem 3. Let $\tilde{a}_j = (a_j, b_j)$ $(j = 1, 2 \cdots, n)$ be a collection of the IFVs, then, the result aggregated from Definition 7 is still an IFV, and even

$$IFGHOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$

$$= \left(\frac{\gamma \left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j} \right)^{1/\lambda}}{\left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} + (\gamma^2 - 1) \prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j} \right)^{1/\lambda} + (\gamma - 1) \left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j} \right)^{1/\lambda}}, \right.$$
$$\left. \frac{\left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} + (\gamma^2 - 1) \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j} \right)^{1/\lambda} - \left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j} \right)^{1/\lambda}}{\left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} + (\gamma^2 - 1) \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j} \right)^{1/\lambda} + (\gamma - 1) \left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j} \right)^{1/\lambda}} \right) \tag{31}$$

where $x_j = \left(1 + (\gamma - 1)(1 - a_j)\right)^{\lambda} + (\gamma^2 - 1) a_j^{\lambda}$, $y_j = \left(1 + (\gamma - 1)(1 - a_j)\right)^{\lambda} - a_j^{\lambda}$ $z_j = \left(1 + (\gamma - 1) b_j\right)^{\lambda} + (\gamma^2 - 1)(1 - b_j)^{\lambda}$, $t_j = \left(1 + (\gamma - 1) b_j\right)^{\lambda} - (1 - b_j)^{\lambda}$, here $\gamma > 0$.

$(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for any j.

An important characteristic of the $IFGHOWA$ operator is that it can weigh the input data according to these data's position in ranking from largest to smallest. So, ω can also be called the position weighted vector.

In general, the position weighted vector ω can be determined by decision makers according to actual needs of decision making problems. In some special cases, it can also be determined by some mathematical methods. Xu (2005) proposed a method shown as follows

$$\omega_j = \frac{e^{-\frac{(j-m_{n-1})^2}{2o_{n-1}^2}}}{\sum_{k=1}^{n-1} e^{-\frac{(j-m_{n-1})^2}{2o_{n-1}^2}}} \quad (j = 1, 2, \cdots, n-1) \tag{32}$$

Where, m_{n-1} and o_{n-1} are the mean and the standard deviation of the collection of $1, 2, \cdots, n-1$, respectively. m_{n-1} and o_{n-1} can be calculated by the following formulas, respectively.

$$m_{n-1} = \frac{n}{2} \tag{33}$$

$$o_{n-1} = \sqrt{\frac{1}{n-1}\sum_{j=1}^{n-1}(j - m_{n-1})^2} \tag{34}$$

Similarly, the *IFGHOWA* operator has also the some properties, such as monotonicity, idempotency, boundedness, and commutativity.

Some special cases of the *IFGHOWA* operator with respect to the parameters λ and γ can be discussed as follows.

(1) If $\lambda = 1$, then the IFGHOWA operator (30) will be reduced to the Hammer intuitionistic fuzzy ordered weighted averaging (HIFOWA) operator. According to (31), we can get

$$HIFOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$

$$= \left(\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)a_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}(1-a_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)a_{\sigma(j)}\right)^{\omega_j} + (\gamma-1)\prod_{j=1}^{n}(1-a_{\sigma(j)})^{\omega_j}}, \frac{\gamma\prod_{j=1}^{n} b_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(1-b_{\sigma(j)})\right)^{\omega_j} + (\gamma-1)\prod_{j=1}^{n} b_{\sigma(j)}^{\omega_j}} \right) \tag{35}$$

Further,

(i) When $\gamma = 1$, the formula (35) will be reduced to an intuitionistic fuzzy ordered weighted averaging (IFOWA) operator which is defined by Xu (2007). It is shown as follows:

$$IFOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \left(1 - \prod_{j=1}^{n}(1-a_{\sigma(j)})^{\omega_j}, \prod_{j=1}^{n} b_{\sigma(j)}^{\omega_j}\right).$$

(ii) When $\gamma = 2$, the formula (35) will be reduced to the Einstein intuitionistic fuzzy ordered weighted averaging (EIFOWA) operator. It is shown as follows:

$$EIFOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\frac{\prod_{j=1}^{n}\left(1+a_{\sigma(j)}\right)^{\omega_j}-\prod_{j=1}^{n}(1-a_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}\left(1+a_{\sigma(j)}\right)^{\omega_j}+\prod_{j=1}^{n}(1-a_{\sigma(j)})^{\omega_j}},\frac{2\prod_{j=1}^{n}b_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}\left(2-b_{\sigma(j)}\right)^{\omega_j}+\prod_{j=1}^{n}b_{\sigma(j)}^{\omega_j}}\right)$$

(2) If $\lambda\to 0$, then the IFGHOWA operator (30) will be reduced to the Hammer intuitionistic fuzzy ordered weighted geometric (HIFOWG) operator. According to (31), we can get

$HIFOWG(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$

$$=\left(\frac{\gamma\prod_{j=1}^{n}a_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)(1-a_{\sigma(j)})\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}a_{\sigma(j)}^{\omega_j}},\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)b_{\sigma(j)}\right)^{\omega_j}-\prod_{j=1}^{n}(1-b_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)b_{\sigma(j)}\right)^{\omega_j}+(\gamma-1)\prod_{j=1}^{n}(1-b_{\sigma(j)})^{\omega_j}}\right)\quad(36)$$

Further,

(i) When $\gamma=1$, the formula (36) will be reduced to an intuitionistic fuzzy ordered weighted geometric (IFOWG) operator which is defined by Xu and Yager (2006). It is shown as follows:

$$IFOWG(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\prod_{j=1}^{n}a_{\sigma(j)}^{\omega_j},1-\prod_{j=1}^{n}(1-b_{\sigma(j)})^{\omega_j}\right).$$

(ii) When $\gamma=2$, the formula (36) will be reduced to the Einstein intuitionistic fuzzy ordered weighted geometric (EIFOWG) operator. It is shown as follows:

$$EIFOWG(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\left(\frac{2\prod_{j=1}^{n}a_{\sigma(j)}^{\omega_j}}{\prod_{j=1}^{n}\left(2-a_{\sigma(j)}\right)^{\omega_j}+\prod_{j=1}^{n}a_{\sigma(j)}^{\omega_j}},\frac{\prod_{j=1}^{n}\left(1+b_{\sigma(j)}\right)^{\omega_j}-\prod_{j=1}^{n}(1-b_{\sigma(j)})^{\omega_j}}{\prod_{j=1}^{n}\left(1+b_{\sigma(j)}\right)^{\omega_j}+\prod_{j=1}^{n}(1-b_{\sigma(j)})^{\omega_j}}\right)$$

(3) If $\gamma=1$, then the IFGHOWA operator (30) will be reduced to the generalized intuitionistic fuzzy ordered weighted averaging (GIFOWA) operator which is defined by Zhao et al. (2010). According to (31), we can get

$GIFOWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$

$$=\left(\left(1-\prod_{j=1}^{n}\left(1-a_{\sigma(j)}^{\lambda}\right)^{\omega_j}\right)^{1/\lambda},1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-b_{\sigma(j)}\right)^{\lambda}\right)^{\omega_j}\right)^{1/\lambda}\right)\quad(37)$$

(4) If $\gamma = 2$, then the IFGHOWA operator (30) will be reduced to the generalized Einstein intuitionistic fuzzy ordered weighted averaging (GEIFOWA) operator. According to (31), we can get

$$IFGHOWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$
$$= \left(\frac{2\left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} + 3\prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} x_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} y_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}, \frac{\left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} + 3\prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} - \left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} + 3\prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda} + \left(\prod_{j=1}^{n} z_{\sigma(j)}^{\omega_j} - \prod_{j=1}^{n} t_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}} \right) \quad (38)$$

Where $x_j = \left(2 - a_j\right)^{\lambda} + 3a_j^{\lambda}$, $y_j = \left(2 - a_j\right)^{\lambda} - a_j^{\lambda}$, $z_j = \left(1 + b_j\right)^{\lambda} + 3(1 - b_j)^{\lambda}$, $t_j = \left(1 + b_j\right)^{\lambda} - (1 - b_j)^{\lambda}$.

As $IFGHWA$ and $IFGHOWA$ operators emphasize the self-importance and the position importance of each IFV, respectively. However, in general, we need consider these two aspects together because they represent different points of decision making problems. In order to overcome the shortcomings, a generalized hybrid averaging operator based on Hamacher operations is given as follows.

Definition 8. Let $\tilde{a}_j = \left(a_j, b_j\right)$ $(j = 1, 2 \cdots, n)$ be a collection of the IFVs, and $IFGHHWA: \Omega^n \to \Omega$, if

$$IFGHHWA(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \bigoplus_{h\, j=1}^{n} \left(\omega_j \tilde{b}_{\sigma(j)}\right) \quad (39)$$

Where Ω is the set of all IFVs, and $\omega = \left(\omega_1, \omega_2, \cdots, \omega_n\right)^T$ is the weighted vector associated with $IFGHHWA$, such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. $w = (w_1, w_2, \cdots, w_n)$ is the weight vector of $\tilde{a}_j (j = 1, 2, \cdots, n)$, and $w_j \in [0,1], \sum_{j=1}^{n} w_j = 1$. Let $\tilde{b}_j = n w_j \tilde{a}_j = \left(\dot{a}_j, \dot{b}_j\right)$, n is the adjustment factor. Suppose $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\tilde{b}_{\sigma(j-1)} \geq \tilde{b}_{\sigma(j)}$ for any j, and then function $IFGHHWA$ is called the intuitionistic fuzzy generalized Hamacher hybrid weighted averaging (IFGHHWA) operator.

Based on the Hamacher operational rules of the IVIFNs, we can derive the result shown as theorem 4.

Theorem 4. Let $\tilde{a}_j = \left(a_j, b_j\right)$ $(j = 1, 2 \cdots, n)$ be a collection of the IFVs,then, the result aggregated from Definition 8 is still an IFV, and even

$$IFGHHWA(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$$
$$=\left(\frac{\gamma\left(\prod_{j=1}^{n}\dot{x}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{y}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}\dot{x}_{\sigma(j)}^{\omega_j}+(\gamma^2-1)\prod_{j=1}^{n}\dot{y}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}\dot{x}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{y}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}},\right. \tag{40}$$
$$\left.\frac{\left(\prod_{j=1}^{n}\dot{z}_{\sigma(j)}^{\omega_j}+(\gamma^2-1)\prod_{j=1}^{n}\dot{t}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{n}\dot{z}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{t}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{n}\dot{z}_{\sigma(j)}^{\omega_j}+(\gamma^2-1)\prod_{j=1}^{n}\dot{t}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}+(\gamma-1)\left(\prod_{j=1}^{n}\dot{z}_{\sigma(j)}^{\omega_j}-\prod_{j=1}^{n}\dot{t}_{\sigma(j)}^{\omega_j}\right)^{1/\lambda}}\right)$$

Where $\dot{x}_j=\left(1+(\gamma-1)(1-\dot{a}_j)\right)^{\lambda}+(\gamma^2-1)\dot{a}_j^{\lambda}$, $\dot{y}_j=\left(1+(\gamma-1)(1-\dot{a}_j)\right)^{\lambda}-\dot{a}_j^{\lambda}$
$\dot{z}_j=\left(1+(\gamma-1)\dot{b}_j\right)^{\lambda}+(\gamma^2-1)(1-\dot{b}_j)^{\lambda}$, $\dot{t}_j=\left(1+(\gamma-1)\dot{b}_j\right)^{\lambda}-(1-\dot{b}_j)^{\lambda}$.

$$\dot{a}_j=\frac{\gamma a_j^{nw_j}}{\left(1+(\gamma-1)(1-a_j)\right)^{nw_j}+(\gamma-1)a_j^{nw_j}},\ \dot{b}_j=\frac{\left(1+(\gamma-1)b_j\right)^{nw_j}-(1-b_1)^{nw_j}}{\left(1+(\gamma-1)b_j\right)^{nw_j}+(\gamma-1)(1-b_j)^{nw_j}}$$

$(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$,such that $\tilde{b}_{\sigma(j-1)}\geq\tilde{b}_{\sigma(j)}$ for any j , $\tilde{b}_j=\left(\dot{a}_j,\dot{b}_j\right)=nw_j\tilde{a}_j$.

Theorem 5. The $IFGHWA$ operator and $IFGHOWA$ operator are the special cases of the $IFGHHWA$ operator.

It is easy to prove that when $W=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, the $IFGHHWA$ operator will reduce to $IFGHOWA$ operator, and when $\omega=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, the $IFGHHWA$ operator will reduce to $IFGHWA$ operator.

4. Multiple attribute group decision making methods based on generalized Hamacher aggregation operators

4.1. Description the decision making problems

For a multiple attribute group decision making problem, let $E=\left\{e_1,e_2,\cdots,e_q\right\}$ be the collection of decision makers, $A=\left\{A_1,A_2,\cdots,A_m\right\}$ be the collection of alternatives, and $C=\left\{C_1,C_2,\cdots,C_n\right\}$ be the collection of attributes. Suppose that $\tilde{a}_{ij}^{k}=\left(a_{ij}^{k},b_{ij}^{k}\right)$ is an attribute value given by the decision maker e_k, which it is expressed in IFV for the alternative A_i with respect to the attribute C_j, $w=(w_1,w_2,\cdots,w_n)$ is the weight vector of attribute set $C=\left\{C_1,C_2,\cdots,C_n\right\}$, and $w_j\in[0,1],\sum_{j=1}^{n}w_j=1$. Let $\varpi=(\varpi_1,\varpi_2,\cdots,\varpi_q)$ be the vector of decision makers $\left\{e_1,e_2,\cdots,e_q\right\}$, and $\varpi_k\in[0,1],\sum_{k=1}^{q}\varpi_k=1$. Then we use the attribute weights, the decision makers' weights, and the attribute values to rank the order of the alternatives.

4.2. The methods based on generalized Hamacher hybrid weighted averaging operator

Step 1: Normalize the decision-making information

In general, for attribute values, there are benefit attributes (I_1) (the bigger the attribute values the better) and cost attributes (I_2) (the smaller the attribute values the better). In order to eliminate the impact of different type attribute values, we need normalize the decision-making information. Of course, if all the attributes are of the same type, then they do not need normalization.

We may transform the attribute values from cost type to benefit type, in such a case, decision matrices $A^k=\left[\tilde{a}_{ij}^{k}\right]_{m\times n}(k=1,2,\cdots,q)$ can be transformed into matrices $R^k=\left[\tilde{r}_{ij}^{k}\right]_{m\times n}(k=1,2,\cdots,q)$. where, $\tilde{r}_{ij}^{k}=\left(t_{ij}^{k},f_{ij}^{k}\right)$.

$$\tilde{r}_{ij}^{k}=\tilde{r}_{ij}^{k}=\left(t_{ij}^{k},f_{ij}^{k}\right)=\begin{cases}\left(a_{ij}^{k},b_{ij}^{k}\right)\text{for benefit attribute } C_j\\ \left(b_{ij}^{k},a_{ij}^{k}\right)\text{ for cost attribute } C_j\end{cases}\quad i=1,2,\cdots,m,\ j=1,2,\cdots,n \tag{41}$$

Step 2: Utilize the IFGHHWA operator

$$\tilde{r}_{ij}=\left(t_{ij},f_{ij}\right)=IFGHHWA(\tilde{r}_{ij}^{1},\tilde{r}_{ij}^{2},\cdots,\tilde{r}_{ij}^{q}) \tag{42}$$

to aggregate all the intuitionistic fuzzy decision matrixes $R^k=\left[\tilde{r}_{ij}^{k}\right]_{m\times n}(k=1,2,\cdots,q)$ into the collective intuitionistic fuzzy decision matrix $R=\left[\tilde{r}_{ij}\right]_{m\times n}$.

Step 3: Utilize the IFGHHWA operator

$$\tilde{r}_{i}=\left(t_{i},f_{i}\right)=IFGHHWA(\tilde{r}_{i1},\tilde{r}_{i2},\cdots,\tilde{r}_{in}) \tag{43}$$

to derive the collective overall preference values $\tilde{r}(i=1,2,\cdots,m)$.

Step 4: calculate the score function $S(\tilde{r}_i)(i=1,2,\cdots,m)$ of the collective overall values $\tilde{r}_i(i=1,2,\cdots,m)$, and then rank all the alternatives $\{A_1, A_2, \cdots, A_m\}$. When two score functions $S(\tilde{r}_i)$ and $S(\tilde{r}_j)$ are equal, we need calculate their accuracy functions $H(\tilde{r}_i)$ and $H(\tilde{r}_j)$, then we can rank them by accuracy functions.

Step 5: Rank the alternatives

Rank all the alternatives $\{A_1, A_2, \cdots, A_m\}$ and select the best one(s) by score function $S(\tilde{r}_i)$ and accuracy function $H(\tilde{r}_i)$.

Step 6: End.

5. An application example

In order to demonstrate the application of the proposed methods, we will cite an example about the air quality evaluation (adapted from Yue (2011)). To evaluate the air quality of Guangzhou for the 16th Asian Olympic Games which would be held during November 12–27, 2010. The air quality in Guangzhou for the Novembers of 2006, 2007, 2008 and 2009 were collected in order to find out the trends and to forecast the situation in 2010. There are 3 air-quality monitoring stations (e_1, e_2, e_3) which can be seen as decision makers, and their weight ϖ is $(0.314, 0.355, 0.331)^T$. There are 3 measured indexes, namely, SO_2(C_1), NO_2(C_2) and PM_{10}(C_3), and their weight w is $(0.40, 0.20, 0.40)^T$. The measured values from air-quality monitoring stations under these indexes are shown in Table1, Table 2 and Table 3, and they can be expressed by intuitionistic fuzzy numbers. Let (A_1, A_2, A_3, A_4) ={November of 2006,November of 2007,November of 2008, November of 2009}be the set of alternatives, please give the rank of air quality from 2006 to 2009.

5.1. Rank the alternatives by the proposed method.

To get the best alternative(s), the following steps are involved:

Step 1: Normalize the decision-making information

Because all the measured values are of the same type, then they do not need normalization.

Step 2: Utilize e the IFGHHWA operator expressed by (42) to aggregate all the individual intuitionistic fuzzy decision matrixes $R^k = \left[\tilde{r}_{ij}^k\right]_{4\times3} (k=1,2,3)$ into the collective intuitionistic fuzzy decision matrix $R = \left[\tilde{r}_{ij}\right]_{4\times3}$. We can get (suppose $\lambda = 1, \gamma = 2, \omega = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$)

$$R=\begin{bmatrix}(0.221,0.366) & (0.260,0.322) & (0.282,0.273)\\(0.324,0.365) & (0.353,0.352) & (0.197,0.265)\\(0.310,0.309) & (0.359,0.157) & (0.268,0.213)\\(0.396,0.270) & (0.376,0.277) & (0.402,0.092)\end{bmatrix}$$

Table1. Air quality data from station e_1

	C_1	C_2	C_3
A_1	(0.265,0.385)	(0.330,0.280)	(0.245,0.480)
A_2	(0.345,0.410)	(0.430,0.280)	(0.245,0.380)
A_3	(0.365,0.335)	(0.480,0.205)	(0.340,0.290)
A_4	(0.430,0.270)	(0.460,0.295)	(0.310,0.170)

Table2. Air quality data from station e_2

	C_1	C_2	C_3
A_1	(0.125,0.405)	(0.220,0.360)	(0.345,0.165)
A_2	(0.355,0.330)	(0.300,0.330)	(0.205,0.165)
A_3	(0.315,0.305)	(0.330,0.105)	(0.280,0.200)
A_4	(0.365,0.270)	(0.355,0.325)	(0.425,0.090)

Table3. Air quality data from station e_3

	C_1	C_2	C_3
A_1	(0.260,0.315)	(0.220,0.330)	(0.255,0.245)
A_2	(0.270,0.360)	(0.320,0.465)	(0.135,0.290)
A_3	(0.245,0.290)	(0.250,0.180)	(0.175,0.165)
A_4	(0.390,0.270)	(0.305,0.220)	(0.465,0.050)

Step 3: Utilize the IFGHHWA operator expressed by (43) to derive the collective overall preference values (suppose $\lambda=1$, $\gamma=2$, $\omega=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$), we can get

$$\tilde{r}_1=(0.288,0.305),\tilde{r}_2=(0.329,0.310),\tilde{r}_2=(0.346,0.210),\tilde{r}_4=(0.417,0.183)$$

Step 4: Calculate the score function $S(\tilde{r}_i)(i=1,2,3,4)$ of the collective overall values $\tilde{r}_i(i=1,2,3,4)$, we can get

$$S(\tilde{r}_1)=\text{-}0.017,\ S(\tilde{r}_2)=0.019,\ S(\tilde{r}_3)=0.136,\ S(\tilde{r}_4)=0.234$$

Step 5: Rank the alternatives

According to the score function $S(\tilde{r}_i)(i=1,2,3,4)$, we can get

$A_4 \succ A_3 \succ A_2 \succ A_1$.

So, the best alternative is A_4, i.e., the best air quality in Guangzhou is November of 2009 among the Novembers of 2006, 2007, 2008, and 2009.

5.2. The influence of the parameters λ, γ on the result of this example

In order to illustrate the influence of the parameters λ, γ on decision making of this example, we use the different values λ, γ in steps 2 and 3 to rank the alternatives. We can get the aggregation results using the different aggregation parameters λ and γ are different, but the orderings of the alternatives are the same in this example. In general, we can take the values of the parameter $\lambda = 1$ for arithmetic aggregation operator, or $\lambda \to 0$ for geometric aggregation operator, and $\gamma = 1$ for Algebraic aggregation operator or $\gamma = 2$ for Einstein aggregation operator.

5.3. Compare with the other methods

In order to verify the effective of the proposed methods, we can compare with the methods proposed by Xu (2007), Xu and Yager (2006). Firstly, we use these methods to rank the alternatives in this example, and there are the same ranking results for these methods. Secondly, the aggregation operators proposed by Xu (2007), Xu and Yager (2006) are a special case of the $IFGHHWA$ operator proposed in this paper. So the method presented in this paper is more general and more flexible.

6. Conclusion

In real decision making, the decision making problems are fuzzy and uncertain, and the intuitionistic fuzzy values are easier to express the fuzzy decision information, this paper explored some generalized Hamacher aggregation operators based on IFVs and applied them to the multi-attribute group decision making problems in which attribute values take the form of IFVs. Firstly, intuitionistic fuzzy generalized Hamacher weighted averaging (IFGHWA) operator, intuitionistic fuzzy generalized Hamacher ordered weighted averaging (IFGHOWA) operator, and intuitionistic fuzzy generalized Hamacher hybrid weighted averaging (IFGHHWA) operator, were proposed. They provide very general formulations that include as special cases a wide range of aggregation operators for intuitionistic fuzzy information, including all arithmetic aggregation operators, geometric aggregation operators, the Algebraic aggregation operators and Einstein aggregation operators. So they can easily accommodate the environment in which the given arguments are intuitionistic fuzzy sets. Then some desirable properties of these operators, such as commutativity, idempotency, monotonicity and boundedness, were studied, and some special cases in these operators were analyzed. Furthermore, one method to multi-criteria decision group making based on these operators was developed, and the operational processes were illustrated in detail. Finally, an illustrative example is given to verify the developed approach and to demonstrate its practicality and effectiveness. In further research, it is necessary and meaningful to give the applications of these operators to the other domains such as pattern recognition, fuzzy cluster analysis and uncertain programming, etc.

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