Professor Cristinca FULGA, PhD
Department of Applied Mathematics
The Bucharest University of Economic Studies
E-mail: fulga@csie.ase.ro

INTEGRATED DECISION SUPPORT SYSTEM FOR PORTFOLIO SELECTION WITH ENHANCED BEHAVIORAL CONTENT

Abstract. *The portfolio selection problem is a crucial problem that every investor at individual or institutional level has to deal with. There is a vast amount of literature about systems designed to support portfolio management decisions with a large diversity in focus and approach. However, even if it is well known that all decisions depend on decision maker's preferences, the preferences are not represented satisfactorily in most systems. In this paper, we propose a decision support system design for portfolio selection that relies on an optimization model with enhanced behavioural content based on the stochastic programming paradigm. The proposed system is capable of supporting loss averse investors in the complex task of selecting portfolios that are simultaneously optimal from the reward-risk viewpoint and suitable for investor's specific loss aversion profile.*

Keywords*: Decision support system, loss aversion, risk measure, utility function, portfolio optimization.*

JEL Classification : C02, C61, G11.
Math. Subj. Classification 2010: 90B50, 91G10

1. Introduction

Modern optimization based decision support systems (DSS) endow the users with a wide range of competencies. Having at their core optimization models, they provide a powerful tool: the model formulation allows the mathematical representation of complex decision-making problems thus offering support in many domains of applications. The field of portfolio management has a high level of complexity because of the presence of uncertainty, and also because of the need of analyzing huge amount of data and operating in a short time. Therefore, the use of a DSS makes a high impact on decision-making since it improves significantly the quality of the solutions. Quantitative methodologies enhanced by the recent advances in information technology and computer science ensure this quality.

Decision support systems for portfolio management. There is a vast amount of literature about the portfolio selection problem, see for example systems designed to support portfolio management decisions with a large diversity in focus and approach. Because of the many new challenges arising from a more dynamic business environment and because of the increasing complexity of the decision situations, the traditional DSS have evolved into complex integrated decision support systems. Apart enriching the content of the three basic capability subsystems that characterize any DSS (the data management subsystem, the model management subsystem, and the user interface, see Turban et al., 2005), integrated DSSs have implemented new technologies. Thus, they take into consideration the modern business environment, for a thoroughly state-of-the-art on integrated DSSs, see Liu et al. (2009), Zaraté et al. (2014). Many DSSs are based on Multi-criteria Decision Making (MCDM) techniques. Some interactive multi-criteria DSSs are: *ADELAIS* - Siskos and Despotis (1989), *MINORA* - Siskos et al. (1993), *MARKEX* - Siskos and Matsatsinis (1993), *FINEVA* – Zopounidis et al. (1996), *MUSA* - Siskos et al. (1998), *MIIDAS* - Siskos et al. (1999), *INVESTOR* - Zopounidis and Doumpos (2000*a*), *PREFDIS* - Zopounidis and Doumpos (2000*b*). Some DSSs for portfolio management are focused on specific stages of the investment process. For example, the DSSs presented in Xidonas et al. (2011) and Fulga (2015) are focused on the portfolio optimization stage. The DSS proposed in this paper differs from the others due to the emphasis put on the modeling subsystem: investor's preferences are taken into consideration, thus the DSS has an enhanced behavioral content.

Loss aversion in portfolio selection. The portfolio selection problem is studied from a large variety of viewpoints, and many methods and approaches in which different attitudes toward risk were proposed, see for example Afreen Arif and Pakkala (2015), Rezaie et al. (2015), Fulga (2016*a*, 2016*b*), Borgonovo and Gatti (2013), just to name a few. The presence of loss aversion is well documented in literature: Kahneman and Tversky (1979) and Tversky and Kahneman (1992) have presented evidence that investors consider the deviations of their terminal wealth as gains and losses starting from a reference level, and they react differently to gains than to losses. Loss aversion refers to the fact that investors are more sensitive to losses than to gains, where the critical return level θ separates gains from losses. This threshold θ depends heavily on the investor's perception on his own financial situation and on how he interprets the present economic situation and relates to it. There is a variety of reasons triggered by real-life constraints explaining how one ends up having a critical level θ: some investors might violate lone contractual clauses (covenants) if their assets fall below a specified value; others face regulatory mandates which require a minimum level of reserves. Also, in the practice of risk, when asset levels fall under the critical threshold, fund managers are penalized - see for example Borgonovo and Gatti (2013) who investigated the consequences of including covenant breach in the risk analysis of large industrial projects, and provided an objective view on their effect and significance.

Purpose of this paper. It is well-known that all decisions depend on decision maker's preferences. But, although decisions are frequently influenced by the recommendations of computer-based DSSs, the preferences are not represented satisfactorily in most systems. In this paper we propose a DSS for portfolio selection with enhanced behavioral content which is capable of supporting loss averse investors in the complex task of selecting portfolios that are simultaneously performant from the reward-risk view-point and suitable for their specific risk profile. The remainder of this article is organized as follows: in Section 2 we present the system architecture, and illustrate the main technical choices and the functional modules which implement the proposed methodological steps. In Section 3, we give the *Mean-Risk* model on which the Model Management Subsystem is based. Given investor's loss-averse profile, we argue that the appropriate risk measure used is the Expected shortfall with loss aversion parameter θ, $ESLA^{\theta}$. The theoretical results from Theorems 1 and 2 are of great practical importance: for the case when the targeted expected return value is the critical return level θ, the two theorems show that under some mild assumptions the three models *Mean-Variance*, *Mean-*$ESLA^{\theta}$ and *EU* maximization are equivalent. The use of the *Mean-Variance* model instead of one of the other two results in a significant shortening of computer overall runtime and increase in precision due to the very efficient toolboxes for *Mean-Variance* optimization. Concluding remarks from Section 4 end the paper.

2. Methodological framework and system architecture

Distinctive features of the proposed system related to the loss averse profile of DSS' user.

• One decisive element of the portfolio model that captures investor's loss aversion is the risk measure used in the model. Given that the loss-averse investor's main concern is related to the cases when the portfolio return falls under investor critical level θ, the risk measure used in the proposed system is the *Expected Shortfall with Loss Aversion* parameter θ denoted by $ESLA^{\theta}$ and defined in Fulga (2016*a*). For fixing the ideas, let n be the number of stocks used to build portfolios. The key random inputs in the portfolio problem are the random vector of asset returns denoted by $\boldsymbol{r}(\omega)=\left(r_1(\omega),\ldots,r_n(\omega)\right)^T$, $\omega \in \Omega$, or simply by $\boldsymbol{r}$ (we use bold symbols for vectors). The set Ω represents the set of future states of knowledge and has the mathematical structure of a probability space with a probability measure P for comparing the likelihood of future states ω. Let $R(\boldsymbol{x})=\boldsymbol{x}^T\boldsymbol{r}$ be the return of the portfolio $\boldsymbol{x} \in X$, where X is the set of available portfolios defined as and $\boldsymbol{1} \in \mathbf{R}^n$ is the vector with all components equal to 1. $R(\boldsymbol{x})$ is a $X=\left\{\boldsymbol{x} \in \mathbf{R}^n \,\middle|\, \boldsymbol{x}^T\boldsymbol{1}=1,\ \boldsymbol{x} \geq \boldsymbol{0}\right\}$, random variable having a continuous probability density function (pdf) $g_{R(\boldsymbol{x})}(r)$, $r \in \mathbf{R}$, induced by that of $\boldsymbol{r}$. The probability of $R(\boldsymbol{x})$ not exceeding r is given by the cumulative distribution function (cdf)

$G_{R(x)}(r) = P\{R(x) \le r\}$ which is assumed to be continuous with respect to r. The definition of the risk measure $ESLA^{\theta}$ of the return $R(\boldsymbol{x})$ of a feasible portfolio $\boldsymbol{x}$ is as follows: $ESLA^{\theta}(R(\boldsymbol{x})) = -E(R(\boldsymbol{x}) \mid R(\boldsymbol{x}) \le \theta)$, where it is assumed that $P\{R(\boldsymbol{x}) \le \theta\} > 0$, see Fulga (2016*a*).

• In the proposed DSS, the investment opportunity set from which the loss averse investor choses investor preferred portfolio is the efficient frontier obtained from solving the bi-criterial problem:

$$\min_{\boldsymbol{x} \in X} \left(-E(R(\boldsymbol{x})), ESLA^{\theta}(R(\boldsymbol{x}))\right). \tag{1}$$

which is referred to as *Mean-ESLA*$^{\theta}$ model.

• The method used to select one single preferred efficient portfolio from the entire *Mean-ESLA*$^{\theta}$ efficient frontier relies on the utility function capturing investor's loss aversion. Investor utility function is characterized by two parameters, the critical return level θ, and the coefficient of loss aversion λ used to capture the fact that losses are more painful than equivalent gains (even when the threshold θ is only slightly exceeded). In this paper, we consider an investor characterized by the logarithmic utility function with loss aversion

$$V(r) = \ln(1 + r) - \lambda\left[\theta - r\right]^{+}, \; r \in \mathbf{R}, \tag{2}$$

where $\left[a\right]^{+} = \max\{0, a\}$.

We determine the preferred efficient portfolio (called *optimal*) based on the following procedure: firstly, we calculate the expected utility of returns $E(V(R(\boldsymbol{x})))$ for all efficient portfolios $\boldsymbol{x}$, and then we select the portfolio with the highest expected utility value.

• Besides the elements related to the model, the system supports the user by having implemented many options in all subsystems: thus, investor' loss aversion is fully taken into consideration in all phases.

Decision support system architecture. The proposed system consists of the following components:

(1) *The Graphical User Interface* is a user-friendly module that enables the user to handle easily the input and the output data, and exploit the optimization toolboxes. The windows and menus allow that the complexities of the system core to remain hidden to the user.

(2) *The Data Management Subsystem*. The functions performed are as follows:

(2a) *Online connection to data servers*. Currently, the server used is Yahoo! Financial for US stocks. The program is connected online to the server, retrieving permanently new data.

(2b) *Packing data in high level objects*. The primary data is structured in objects,

ready to be used. For example, the downloaded data are organized in time series objects, made accessible by MATLAB environment through specific Toolboxes.
(3) *The Model Management Subsystem.*
(3a) *The objective functions*: the expected return of the portfolio $E\left(R\left(x\right)\right)$ and the risk measure $ESLA^{q}\left(R\left(x\right)\right)$.
(3b) *The constraints*. The model can incorporate several types of pre-defined constraints for portfolio assets such as linear and/or nonlinear equalities and inequalities, lower and upper bounds, budget constraints, group, group ratio and turnover constraints. The most common constraints encountered, mandatory for portfolio models are:
- *Budget constraint*,
- *Diversification constraint,*
- *Group constraints* allowing specific sector preferences,
- *Group ratio constraints* allowing a specific ratio between two groups,
- *Lower and upper bound constraints* on portfolio weights, including short selling,
- *Capitalization adjustment*,
- *Average Turnover Constraints* – linear absolute value constraint that enforces an upper bound on the average purchase/ sells.
(3c*) Investor preferences* are captured in the *Mean-ESLA*$^{\theta}$ model and in the logarithmic utility function with loss aversion based on which the final choice is made.
(3d) *Scenario generation*. In order to capture a broad specter of preferences, several methods for scenario generation are implemented in the proposed DSS. The method used to generate paths of future asset returns in our computational results is the Filtered Historical Simulation method briefly described in the sequel. For each of the *n* assets, we use a combination of ARMA(1,1) for the conditional mean and GARCH(1,1) for volatility. The standard residuals obtained from the application of the econometric model are filtered to generate a series of independent identically distributed standardized (i.i.d.) residuals and then bootstrapped.
(3e) *Portfolio selection methodology.* Firstly, the *Mean-ESLA*$^{\theta}$ efficient frontier is determined, and secondly the efficient portfolio with the highest expected utility value is selected as optimal solution.

The proposed DSS differs from other systems by the following special contributions:
(i) The system core relies on innovative methodologies for the portfolio selection problem and takes advantage of the computational power offered by high-performance computing environment. This feature is particularly important for applications for which a huge amount of data should be stored, managed and analyzed.

(*ii*) During the critical phase of formulating the investment policy statement (and expressing investor investment objectives and constraints), the proposed DSS fully supports investor to model the preferences accurately.
(*iii*) Due to its modular structure, the system can be extended to other problems that uses similar methodologies.
(*iv*) The user can access the system by a user- friendly interface and modify parameters/settings according specific needs/preferences.
(*v*) The optimization module is developed in MATLAB highly technical environment using object oriented programming. Thus, the objects, data structures, methods developed in specific toolboxes can be used as they are or they can be further developed to respond better to new demands. It uses advances in database structures, internet technology, client-server architecture, and cloud computing that are specific features of integrated DSS.
The flowchart of the proposed methodology is graphically depicted in Figure 1.

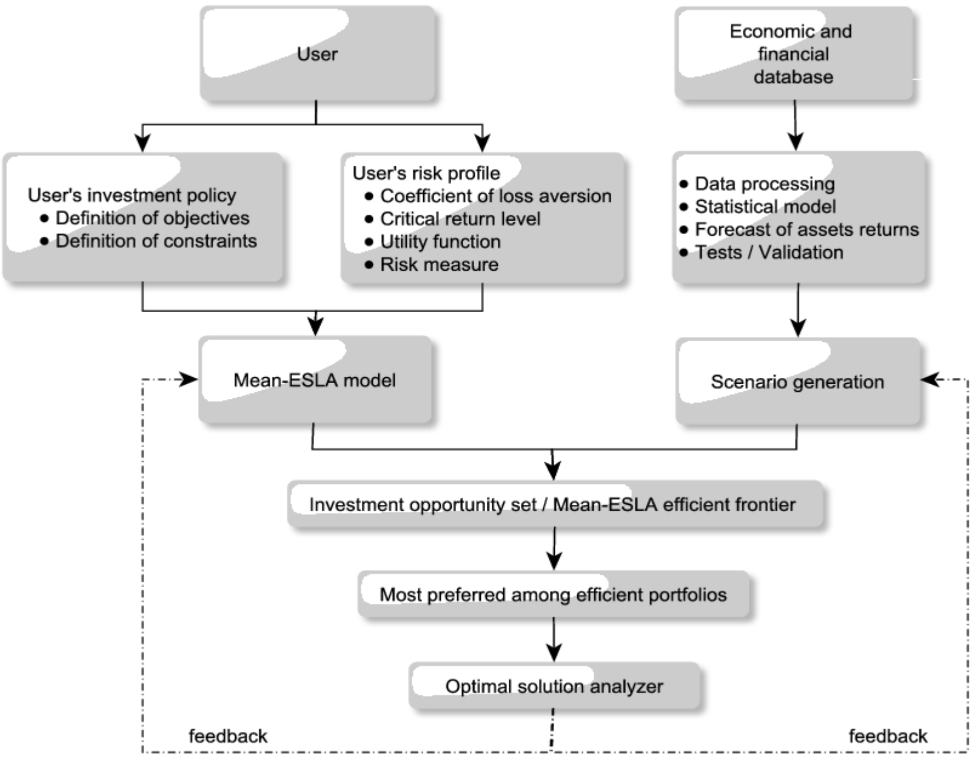

Figure 1. **Process flowchart of the proposed methodology.**

3. Portfolio selection with loss aversion

The model management subsystem is based on the *Mean-ESLA* $^{\theta}$ model (1). The first step is the construction of the *Mean-ESLA* $^{\theta}$ efficient frontier. This relies on solving the minimum risk model

$$\min_{x \in X_{\gamma}} ESLA^{\theta}\left(R\left(x\right)\right), \tag{3}$$

where $X_{\gamma} = \left\{x \in X \,\middle|\, E\left(R\left(x\right)\right) = \gamma\right\}$. We shall refer to this model as (*ME*).

Varying γ in the range of the expected return, we obtain the *Mean-ESLA*$^{\theta}$ efficient frontier. The second step of selecting one single preferred portfolio out of the frontier relies on the utility function (2) that captures investor's loss aversion.

3.1 Method used to elicit investor's utility function with loss aversion

There are several possible methods to elicit investor's utility function. The most frequently used class of methods relies on laboratory experiments aiming to determine points of the utility function. There are various possible procedures for determining them. The general approach is as follows: the investor is asked to express his preferences for monetary gains and losses in several simple and hypothetical lottery situations. It is obviously inferred that the preferences expressed in the hypothetical setting will carry over to much more complex real world decision problems, including portfolio selection problems. The responses provided are then utilized to empirically estimate the individual's utility function: a fit of a function can be made, where usually a specific functional form is assumed, in our case (1), or an affine transformation of it. This method has a couple of advantages for the investors untrained in decision making under uncertainty: it uses simple lotteries that do not involve unintuitive probabilities. Moreover, it only needs relatively few questions to elicit a utility function.

Experimental study. For determining the parameters of the utility function *V*, we use the *midpoint certainty equivalent method*, see for example Hens and Rieger (2010). The steps taken are the following:

Step 1. We determine the range of expected returns for *Mean-ESLA* $^{\theta}$ efficient frontier: $\left[E\left(R\right)_{\min}; E\left(R\right)^{\max}\right]$ where $E\left(R\right)_{\min}$ is the expected return of the global minimum risk portfolio, and $E\left(R\right)^{\max}$ is the solution of the maximization problem $\max_{x \in X} E\left(R\left(x\right)\right)$.

Step 2. We set $V\left(E\left(R\right)_{\min}\right) = -10,\ V\left(E\left(R\right)^{\max}\right) = 1,$ which is allowed because the utility function V is determined up to an affine transformation.

Step 3. In the iteration step, the investor provides the certainty equivalent CE_1 of the lottery $\tilde{L}_1$ with the outcomes $E\left(R\right)_{\min}$ and $E\left(R\right)^{\max}$ that each occur with probability ½. We find the utility value $V\left(CE_1\right)$ by applying the definition of the certainty equivalent:

$$V\left(CE_1\right) = E\left(V\left(\tilde{L}_1\right)\right) = V\left(E\left(R\right)_{\min}\right) \times \frac{1}{2} + V\left(E\left(R\right)^{\max}\right) \times \frac{1}{2} = -4.5.$$

The investor provides the certainty equivalent CE_2 of the lottery $\tilde{L}_2$ constructed with the outcomes $E\left(R\right)_{\min}$ and CE_1 (or CE_1 and $E\left(R\right)^{\max}$) both occurring with probability ½. We calculate analogously the value $V(CE_2)$:

$$V\left(CE_2\right) = E\left(V\left(\tilde{L}_2\right)\right) = V\left(E\left(R\right)_{\min}\right) \times \frac{1}{2} + V\left(CE_1\right) \times \frac{1}{2} = -7.25.$$

Up to now, we have four data points $P_1\left(E\left(R\right)_{\min}, -10\right)$, $P_2\left(E\left(R\right)^{\max}, +1\right)$, $P_3\left(CE_1, -4.5\right)$, and $P_4\left(CE_2, -7.25\right)$ that will be used to determine the utility function. Repeating these iterations we obtain more data points; in total, we have constructed 24 points $P_1,\ldots,P_{24}$. These computations ultimately lead to the representation of the utility function by points. After fitting the utility function of the form $a\left(\ln\left(1+r\right) - \lambda\left[\theta - r\right]^{+}\right) + b,$ where $a > 0,$ and $b \in \mathbf{R}$, on the points obtained $P_1,\ldots,P_{24},$ we find the parameters values a, b, λ, and θ. Thus, we have found investor's utility function with loss aversion $V\left(r\right) = \ln\left(1+r\right) - \lambda\left[\theta - r\right]^{+},$ with $\lambda = 10,$ and $\theta = 10 \times 10^{-3},$ that is represented graphically in Figure 2 together with the points $P_1,\ldots,P_{24}$ that lead to it.

Repeating these iterations we obtain more data points – we have constructed 24 points. These computations ultimately lead to the representation of the utility function by points as seen in Figure 2.

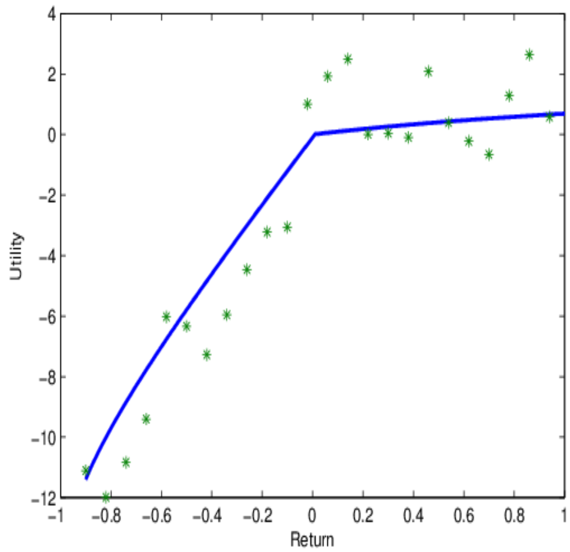

Figure 2. Graphical representation of the logarithmic utility function

3.2 Expected utility maximization under logarithmic utility with loss aversion

Next, we consider the expected utility maximization model for an investor characterized by the logarithmic utility function with loss aversion *V* defined in (2). The values *r* of the portfolio return *R(x)* are usually small, around zero, and $r > -1$, therefore we can use the approximation of second order $\ln(1+r) \approx r - r^2/2$ and work with $U(r) = r - r^2/2 - \lambda\left[\theta - r\right]^+$ which approximates the original utility *V*. The approximated utility of the portfolio return is

$U(R(x)) = R(x) - R^2(x)/2 - \lambda\left[\theta - R(x)\right]^+$. We calculate its expectation:

$$E(U(R(x))) = E(R(x)) - E^2(R(x))/2 - \sigma^2(R(x))/2 - \lambda E\left(\left[\theta - R(x)\right]^+\right),$$

where

$$E\left(\left[\theta - R(x)\right]^+\right) = \theta G_{R(x)}(\theta) - \int_{-\infty}^{\theta} r g_{R(x)}(r)dr = \alpha_x^\theta\left(\theta + ESLA^\theta(R(x))\right).$$

(2a)

In this case, the approximated expected utility (*AEU*) model for a given level γ of the expected portfolio return

$$\max_{x \in X_\gamma} E\left(U\left(R\left(x\right)\right)\right) \tag{3}$$

has the particular form:

$$\max_{x \in X_\gamma} \left\{E\left(R\left(x\right)\right) - E^2\left(R\left(x\right)\right)/2 - \sigma^2\left(R\left(x\right)\right)/2 - \lambda \alpha_x^\theta\left(\theta + ESLA^\theta\left(R\left(x\right)\right)\right)\right\}. \tag{3a}$$

3.3 The case of normally distributed returns

Markowitz' (1952) *Mean-Variance* model at a given level $\gamma > -1$ of $E(R(\boldsymbol{x}))$ is referred to as (*MV*) and is defined as

$$\min_{x \in X_\gamma} \sigma^2\left(R\left(x\right)\right). \tag{4}$$

Next, we assume the vector of returns $\boldsymbol{r}$ normally distributed with vector of means $\boldsymbol{\mu}$ and covariance matrix Σ. Thus, the return $R(\boldsymbol{x})$ is normally distributed, $R\left(\boldsymbol{x}\right) \sim N\left(\mu_{\boldsymbol{x}}, \sigma_{\boldsymbol{x}}\right)$, where $\sigma^2\left(R\left(\boldsymbol{x}\right)\right) = \boldsymbol{x}^T \boldsymbol{\Sigma x}$ is the portfolio variance and $\boldsymbol{\Sigma} = \left(\Sigma_{ij}\right)_{1 \le i, j \le n}$, $\Sigma_{ij} = cov\left(r_i, r_j\right)$, $i, j = \overline{1, n}$.

Theorem 1. Let $\boldsymbol{r} \sim N\left(\boldsymbol{\mu}, \Sigma\right)$. Then, the models (*ME*), (*AEU*), and (*MV*) considered at the same expected return level $\gamma = \theta$ are equivalent in the sense that they provide the same optimal solution.

Remark. Results in Theorem 1 are based on the Taylor approximation of second order of ln(1+*r*) around zero. This approximation performs well only if the realizations of the random variable *R(x)* are close to zero. When working with real data, the values might deviate from zero and thus the accuracy worsens. Numerous empirical studies using empirical distribution and the logarithmic utility have shown that the approximation of $E\left(\ln\left(1 + R\left(x\right)\right)\right)$ by a function of $E\left(R\left(x\right)\right)$ and $\sigma^2\left(R\left(x\right)\right)$ performs better if it is based on the second order approximation of Taylor series around $E\left(R\left(x\right)\right)$, see for example Markowitz (2014), Young and Trent (1969). Let $r > -1$ be a possible value of the portfolio return *R(x)*. The approximation of second order of $L\left(r\right) = \ln\left(1 + r\right)$ around $\mu_{\boldsymbol{x}}$ has the form $L\left(r\right) \approx L\left(\mu_x\right) + L'\left(\mu_x\right)\left(r - \mu_x\right) + \left(r - \mu_x\right)^2 L''\left(\mu_x\right)/2$.

By taking the expectation we find:

$$E\left(L\left(R\left(\boldsymbol{x}\right)\right)\right) \approx \ln\left(1+\mu_x\right)-\frac{1}{2\left(1+\mu_x\right)^2}\sigma_x^2.$$

The utility function approximating the original utility V which uses the second order approximation of Taylor series around the mean will be denoted by $\tilde{U}$. Thus, the model

$$\max_{\boldsymbol{x}\in X_\gamma} E\left(\tilde{U}\left(R\left(\boldsymbol{x}\right)\right)\right) \tag{5}$$

denoted by $\left(AE\tilde{U}\right)$, has the following form:

$$\max_{\boldsymbol{x}\in X_\gamma}\left\{\ln\left(1+E\left(R\left(\boldsymbol{x}\right)\right)\right)-\frac{\sigma_x^2\left(R\left(\boldsymbol{x}\right)\right)}{2\left(1+E\left(R\left(\boldsymbol{x}\right)\right)\right)^2}-\lambda\alpha_x^\theta\left(\theta+ESLA^\theta\left(R\left(\boldsymbol{x}\right)\right)\right)\right\}. \tag{5a}$$

θ	$\theta_1 = 2.66\cdot 10^{-3}$			$\theta_2 = 6\cdot 10^{-3}$			$\theta_3 = 10\cdot 10^{-3}$		
Comp.	*MV*	*ME*	*AEU*	*MV*	*ME*	*AEU*	*MV*	*ME*	*AEU*
x_1	32.98	30.47	32.80	32.97	32.97	32.73	8.65	5.72	7.45
x_2	0.12	0.46	1.15	0.07	0.06	0.68	0.00	0.03	0.03
x_3	14.81	6.71	11.18	0.16	0.15	1.17	0.00	0.00	0.03
x_4	3.37	6.57	6.69	0.14	0.12	0.89	0.00	0.00	0.03
x_5	14.82	11.99	14.65	14.46	14.45	13.23	0.03	4.04	0.26
x_6	12.22	19.61	10.70	21.18	21.28	14.57	28.03	27.38	29.82
x_7	13.19	10.13	13.19	28.44	28.40	29.98	33.00	32.96	32.96
x_8	0.07	0.01	0.68	0.07	0.06	0.72	0.01	0.00	0.03
x_9	0.24	0.02	1.06	1.87	1.86	4.15	30.28	29.68	29.33
x_{10}	8.18	14.03	7.90	0.64	0.64	1.88	0.01	0.18	0.06

Table 1. Optimal portfolios weights expressed as percentages for three critical return values

Theorem 2. Let $\boldsymbol{r} \sim N\left(\boldsymbol{\mu},\Sigma\right)$. Then, the models (*ME*), $\left(AE\tilde{U}\right)$, and (*MV*) considered at the same expected return level $\gamma = \theta$ are equivalent in the sense that they provide the same optimal solution.

3.4 Computational results

To illustrate the equivalence between the three models under the specified conditions, we have chosen ten assets from the New York Stock Exchange whose returns distributions are close to the normal distribution. We have considered the case of the loss-averse investor characterized by the logarithmic utility with loss aversion (1) with $\lambda = 10$, and three critical return thresholds. As expected, the compositions of the corresponding (*ME*), (*AEU*) and (*MV*) optimal portfolios are very similar (Theorem 1). The exact compositions expressed as percentages are presented in Table 1. The small differences observed in Table 1 are due to the fact that the theoretical results were obtained assuming the vector of returns normally distributed, but the distributions used are based on real data, so they are close to the normal distributions, but not matching exactly.

4. Conclusions

In this paper, we present an integrated methodological approach for portfolio optimal selection. The proposed DSS is focused on incorporation of individual investor's loss aversion profile, but accommodates a great variety of preferences if other class of utility functions are used. It is developed in MATLAB highly technical environment using object-oriented programming. Thus, the objects, data structures, methods developed in specific toolboxes can be used as they are or can be further developed to respond better to possible new demands. The proposed DSS uses advances in database structures, internet technology, client-server architecture, and cloud computing that are specific features of integrated DSS. Regarding future improvements of the proposed approach, system could include the introduction of target risk to guide the decisions proposed by the system and match them with investor's risk profile. Additionally, it is well known that taking into account the opinions of experts might improve the solutions provided by the system, therefore using / selecting a group of experts based on historical performance could be beneficial.

Appendix

Proof of Theorem 1. We note that we can interpret $G_{R(x)}(\theta)$ as a variable confidence level and denote it by $G_{R(x)}(\theta) = \alpha_x^\theta \in [0,1]$. From the hypothesis we have $R(x) \sim N(\mu_x, \sigma_x)$. Therefore, we have

$\alpha_x^\theta = \Phi\left((\theta - \mu_x)/\sigma_x\right) + 1/2$, where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{1}{2}t^2} dt$, $\forall z \in \mathbf{R}$. Moreover, let g_Z be the pdf and G_Z the cdf of $Z \sim N(0;1)$. Thus, we have

$ESLA^{\theta}(R(\boldsymbol{x})) = -\mu_{\boldsymbol{x}} + \sigma_{\boldsymbol{x}} g_Z\left(q\left(\alpha_{\boldsymbol{x}}^{\theta}, Z\right)\right) \big/ \alpha_{\boldsymbol{x}}^{\theta}$, where $q\left(\alpha_{\boldsymbol{x}}^{\theta}, Z\right)$ is the Gaussian $\alpha_{\boldsymbol{x}}^{\theta}$-quantile satisfying the condition $G_Z\left(q\left(\alpha_{\boldsymbol{x}}^{\theta}, Z\right)\right) = \alpha_{\boldsymbol{x}}^{\theta}$. Since $q\left(\alpha_x^{\theta}, Z\right) = G_Z^{-1}\left(\alpha_x^{\theta}\right)$, we have $q\left(\alpha_x^{\theta}, Z\right) = G_Z^{-1}\left(G_{R(x)}(\theta)\right) = (\theta - \mu_{\boldsymbol{x}}) / \sigma_{\boldsymbol{x}}$ and $ESLA^{\theta}(R(\boldsymbol{x})) = -\mu_x + \sigma_x \psi_Z\left((\theta - \mu_x)/\sigma_x\right)$, where $g_Z(z) / G_Z(z)$ is denoted by $\psi_Z(z)$, for $z \in \mathbf{R}$ with $G_Z(z) \neq 0$ and

$$\frac{1}{\alpha_x^{\theta}} g_Z\left((\theta - \mu_x)/\sigma_x\right) = \psi_Z\left((\theta - \mu_x)/\sigma_x\right).$$

Therefore, the model (*ME*) at a given level γ > -1 of the expected portfolio return becomes

$$\min_{x \in X_\gamma} \left\{ -E(R(\boldsymbol{x})) + \sigma(R(\boldsymbol{x})) \cdot \psi_Z\left(\frac{\theta - E(R(\boldsymbol{x}))}{\sigma(R(\boldsymbol{x}))} \right) \right\}.$$

Now the equivalence between the models (*MV*) and (*ME*) is immediate. Indeed, because for all feasible $\boldsymbol{x} \in X_\gamma$, $E(R(\boldsymbol{x})) = \gamma$, and $\gamma = \theta$, we have $ESLA^{\theta}(R(\boldsymbol{x})) = -\theta + \sigma_{\boldsymbol{x}} \psi_Z(0) = -\theta + \sigma_x \sqrt{\dfrac{2}{\pi}}$.

Now it is clear that minimizing $ESLA^{\theta}(R(\boldsymbol{x}))$ over all $\boldsymbol{x} \in X_\gamma$ is equivalent to minimizing the variance $\sigma^2(R(\boldsymbol{x}))$ over the same feasible set, and thus the two models have the same optimal solutions. Inserting the expression of $ESLA^{\theta}(R(\boldsymbol{x}))$ in the expression of $E(U(R(\boldsymbol{x})))$ (2a) we get

$$E(U(R(\boldsymbol{x}))) = \mu_{\boldsymbol{x}} - \mu_x^2/2 - \sigma_x^2/2 - \lambda G_Z\left(\frac{\theta - \mu_x}{\sigma_x}\right)\left(\theta - \mu_x + \sigma_x \psi_Z\left(\frac{\theta - \mu_x}{\sigma_x}\right)\right).$$

Thus, the (*AEU*) model has the form

$$\max_{x \in X_\gamma} \Big\{ E(R(\boldsymbol{x})) - E^2(R(\boldsymbol{x}))/2 - \sigma^2(R(\boldsymbol{x}))/2 - \lambda \sigma(R(\boldsymbol{x})) \times$$
$$\times \left[\left((\theta - \mu_x)/\sigma_x\right) G_Z\left((\theta - \mu_x)/\sigma_x\right) + g_Z\left((\theta - \mu_x)/\sigma_x\right) \right] \Big\}.$$

But $\gamma = \theta$ and, for $\boldsymbol{x} \in X_\gamma$, the expected utility of the portfolio return reduces to

$E\left(U\left(R\left(x\right)\right)\right)=\theta-\theta^2/2-\sigma_x^2/2-\lambda\sigma_x\big/\sqrt{2\pi}$, and then (*AEU*) and (*MV*) are equivalent.

Proof of Theorem 2. For $\boldsymbol{x}\in X_\gamma$, we have:

$$E\left(\tilde{U}\left(R\left(x\right)\right)\right)=\ln\left(1+\gamma\right)-\sigma_x^2\Big/\left(2\left(1+\theta\right)^2\right)-\lambda\sigma_x\left(\frac{\theta-\gamma}{\sigma_x}G_Z\left(\frac{\theta-\gamma}{\sigma_x}\right)+g_Z\left(\frac{\theta-\gamma}{\sigma_x}\right)\right)$$

But $\gamma=\theta$, therefore we get

$$E\left(\tilde{U}\left(R\left(x\right)\right)\right)=\ln\left(1+\theta\right)-\sigma_x^2\Big/\left(2\left(1+\theta\right)^2\right)-\lambda\sigma_x\Big/\sqrt{2\pi}$$

showing the equivalence between $\left(AE\tilde{U}\right)$ and (*MV*).

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