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A PROBABILISTIC APPROACH TO SETTING WEIGHTS IN A WEIGHTED-AVERAGE PRODUCT EVALUATION

Abstract. *This paper presents a procedure that estimates how markets perceive a product. A weighted-average approach is used as a model for the product perception. Since this approach requires weights to be assigned to product features, and the weights are unknown and vary among customers, the procedure focuses on estimating expected weights of the entire market – such weights are representative weights. The estimation is performed by constructing a confidence interval for the weights. Further, a more accurate location of the weights in the interval is proposed, the accuracy being based on additional information regarding inferior products. This information suggests that if a product is inferior, its value must be low. Therefore, the weights from the interval which minimize the value of an inferior product are suggested, the minimization serving as an approximation of what low value means. Since a minimization is involved, its existence and uniqueness is discussed.*

Key words: *Product evaluation, weighted average, expected weights, confidence interval, linear optimization.*

JEL Classification: C60, D10, D70, L15, M11, M20

1. INTRODUCTION

Many companies set up their businesses to generate profit. To make this accomplishment, the companies must naturally have a product to offer that is of value to customers. If this is the case, a rational question arises how to detect what a product should look like so that it represented a value to customers. To answer the question, product value as a term must first be defined, i.e. it must be formulated how to evaluate products, and subsequently, the defined value should be broken down to some major features it is formed of. A more subtle decomposition of a product as a whole then sheds more light on what part of it should be altered so that it had a good value and was more appealing to the market.

There is an unavoidable obstacle on the way to becoming a favourite with the customers, however. It is the fact that every customer is different, and has different perceptions regarding products. This automatically implies that it is impossible for any product maker to fully satisfy the entire market. Does it mean there is no point in trying to achieve something that is not achievable? Of course not. We don't really have to be interested in every single customer, we might rather take an interest in the stance of the entire market, and look for a stance that is *typical* of that market. What is typical and what isn't has to be unambiguously defined, of course, and should comply with the intuitive notion that it represents a certain standard a larger part of the market follows. Since characteristics that bear the property of reflecting a typical behaviour exist in the theory of probability, a probabilistic approach offers itself for solving the problem of an appropriate product design. What supports the idea of using the theory of probability, as well, is the fact that finding what is typical in the market requires necessarily that the behaviour of the entire market is known in the first place. This is usually not possible in the markets numbering thousands of customers, and so the only way of getting the idea of what is typical is using a random sample of customers and the principles of statistical inference. Point and interval estimations are among the tools provided by the theory of probability. The technique of confidence intervals is often preferred, since it attaches a specific probability to the location of the sought typical phenomenon.

The stochastic approach alone, however, may not suffice for the aforementioned purposes. Imagining that a confidence interval was constructed for what is desired by a product maker to be known about the typical behaviour of the market, the fact that it is an interval means there is an infinite number of possibilities of how the typical behaviour could look like. This, of course, creates a problem. Nevertheless, this problem maybe tackled, as well, if the product maker selects or estimates such a typical market behaviour contained in the constructed interval that corresponds to its own product value or its own position in the market. If its market position is not good, it may expect the typical market behaviour to be such that yields a low value of its product. The inferior market position may serve as additional information for determining the typical market behaviour.

Several problems have just been outlined. This paper deals with these problems, and presents an approach to them in a general way. To be more specific, the paper deals with the following:

It first defines the term product value as a weighted average of product features. This is a natural approach to evaluating products, taking into account both the features, or what makes up the product, together with the current level of the features, and also the importance with which the features contribute to the overall product value. The importance of the features is expressed by weights. The weighted average is a function of the weights, and it is the weights that product makers are interested in. Product makers would like to know typical weights customers assign to product features, so that it was understood how the entire market perceives the products.

It defines the term typical weight as expected weight, i.e. expected value, if it exists, is used as a description of the typical. Further, a confidence interval, based on weights gathered through a random sampling of customers, is constructed for the vector of expected values that represents the set of typical market weights. It must be stressed that the construction of the interval itself may not be a straightforward task, especially when little is known about the market. In this part of the paper, a general approach is used, based on the Markov's inequality.

Once a confidence interval for the weights is constructed, reflecting a typical perception of product features by the market, the set of typical weights from the interval that minimizes the weighted average, i.e. the overall value, of a given inferior product is derived. This is to be the set of weights that corresponds to those product makers who are not very competitive. This set of weights may be used by these companies as a model for how the market perceives their and other typologically identical products. We shall be interested in the existence and uniqueness, in particular, of the solution that minimizes the weighted average. The reasoning behind selecting the minimization as a model is the following: although there exists one and only one set of expected values for the weights, all that is known is that the vector lies with a high probability in an interval, and another piece of information must be used to locate the vector more precisely. Since noncompetitive producers make products of low value, the minimization may be used as an approximation of the low value and, at the same time, as the other piece of information for detecting where the vector of expected weights might be located. The approach is later demonstrated in an example.

Before proceeding, let us note that different techniques to set weights were proposed in the past by many authors, using various mathematical tools. In the 1970s, Pekelman and Sen (1974) analysed a possibility of modelling customers' behaviour with mathematical programming techniques, although their procedures were limited to market segments of a specific type. Other scientific papers devoted to this subject dealt with weights which were clearly defined by the decision-making body. This is not exactly the kind of situation handled in this paper in the sense that the principles utilized in the paper stress the existence of natural diversity of an entity – weights of decision makers or customers in this case. Further, many articles looked at the problem from the point of view of decision makers. The presented paper views the problem from the point of view of another side – product makers in this case. To give some examples of the papers devoted to weights of decision makers, Nutt (1980), for instance, compared several ways of determining weights of decision makers. Choo and Wedley (1985) used the linear programming methodology and decision makers' past rulings to find proper weights for the decision makers. Other literary examples include Solymosi and Dombi (1986) who worked out a way of setting weights which required information from decision makers in the form of inequalities. From more recent papers, Choo, Schoner and Wedley (1999) were preoccupied with pitfalls of decision-making weights interpretation, giving recommendations to decision makers how to proceed when defining their own weights. A step towards working with weights without prior information from decision makers appeared in a hybrid approach introduced by Ma, Fan and Huang (1999). The approach combines objective information and subjective knowledge to determine weights; More recently, the scientific progress in the field of multicriteria decision-making has focused on modelling uncertainty embedded in the lack of information with fuzzy sets and numbers. Wang, Li and Wang (2009) presented such approach, their procedures requiring some information from decision makers in the form of inequalities. Luo et al. (2009) presented an approach with known weight information, the approach being based on weighted correlation coefficients in an interval-valued intuitionistic fuzzy environment. A similar, but improved approach applied to group decision-making was proposed by Park et al. (2009). Finally, a fuzzy approach based on interval-valued intuitionistic fuzzy decision matrix, was introduced by Ye (2010).

As outlined, many papers were preoccupied with the problem how to help decision makers set their own weights, with the more recent papers dealing with modelling uncertainty of the weights with the fuzzy set theory. The presented paper assumes that customers, or decision makers who decide what products to buy, know what importance they assign to each product feature, and it is the uncertainty embedded in the different perception of the product by each customer that is accentuated in this article.

2. WEIGHTED-AVERAGE APPROACH

We are interested in how customers assess products so that their desires may be met to a greater extent by corresponding responsive product improvements. The approach to product assessment by customer we are going to adopt is based on calculation of a weighted-average $h_1w_1 + h_2w_2 + ... + h_nw_n$, where w_i's are weights, i.e. a set of nonnegative numbers the sum of which equals one, and each h_i is a quantitative expression of the level of the *i*-th product feature observed [4]. It is assumed the levels are already dimensionless so that the summation makes sense. Various techniques exist which achieve this goal, including the so-called normalization and standardization techniques. The strength of the weighted-average approach is clear: in a group of typologically identical and mutually competing products, each product is evaluated by customer with a single number representing an aggregation of the product information. Since there is only one number that is assigned to each product, the products can be arranged from best to worst. Although not the only one, this approach is natural and reflects importance of all the criteria considered. This type of aggregate product evaluation will be utilized in the paper, and will be considered a *model* that describes the decision-making process customers go through when buying a product. It is eventually the customer that defines what features products should possess or how important these features are, but it is product makers that determine the levels h_i's when designing products. The features discussed may include such characteristics as product price or product power efficiency, for instance. Product makers want to recognize the unknown weights, or the importance of the product features, so that they can adjust the levels h_i's to boost the product value.

We shall consider each weight of each customer to be strictly positive and smaller than one to avoid situations when a product feature with no importance to customer is involved in the discussion, or when a product would unnaturally have only one feature of interest. In the next section, we shall also presume that a random sample has been drawn from a population of customers. Each weight will then be considered a random variable whose sampled values are available as a result of the random drawing from a probability distribution which may be approximated by a continuous distribution. Therefore, each weight will be considered a continuous random variable.

Regarding the h_i's, although they are often positive in the real world, we do not restrict their values to any set. An adverse yield provided by a financial product may serve as an example of h_i being negative. We will, however, assume that $h_i \neq h_j$ for $i \neq j$. This condition will not affect the existence of a solution we will seek in relation to the unknown weights, but will have an effect on the uniqueness of the solution.

3. INTERVAL FOR WEIGHTS

In this section, we shall construct a general confidence interval for the vector of expected values of the weights used by customers to evaluate products of a kind. The expected value of a weight is considered to be what we previously referred to as a typical weight, and the motivation behind using this characteristic is the fact that each customer simply assigns different importance or weight to a product feature, generally speaking. Thus, a set of weights that is representative of the market's product perception is needed. Of course, when talking about confidence intervals, we automatically put ourselves in the framework of the probability theory. This is a reasonable model to work with here because a customer base may be looked at as a population and its customers' sets of weights in the product evaluation may be viewed as all possible realizations of a random vector, especially when a customer random poll is run to get the idea about the market.

Let us assume a random selection of *m* customers from the customer base has been realized. Each of the customers was asked to provide their individual set of weights for a selected product. Their answers have the form of

$$(w_1^{(i)}, w_2^{(i)}, ..., w_n^{(i)}),\ \ i = 1, 2, ..., m. \tag{1}$$

These sets of values represent realizations of $m \cdot n$ random variables $W_1^{(i)}, W_2^{(i)}, ..., W_n^{(i)},\ i = 1, 2, ..., m,$ the probability distributions of which are unknown. What is known is that, for a given $k, 1 \le k \le n,$ the variables $W_k^{(i)}, i = 1, 2, ..., m,$ represent a random sample drawn from the same probability distribution. As outlined in the previous section, we also assume that the distribution is, at least approximately within our model, continuous. Thus, we work with *n* continuous probability distributions. Althouth not always ideal, a normal distribution may be visualized as a model. In this context, we also assume that all the random variables are positive and smaller than one on the probability space (Ω, S, P) where they are defined. Since $0 \le \left[W_k^{(i)}\right]^l \le 1,$ where $\left[W_k^{(i)}\right]^l$ is the *l*-th power of $W_k^{(i)}$, and this is true for each *k* and *i* considered and each positive integer *l*, all moments of all the variables exist as finite numbers [3], including the expected values $\mu_k = E(W_k^{(i)}),\ k = 1, 2, ..., n,\ i = 1, 2, ..., m,$ and variances $\sigma_k^2 = \mathrm{var}(W_k^{(i)}),\ k = 1, 2, ..., n,\ i = 1, 2, ..., m.$ This means that $\overline{W_k} = m^{-1}\sum_{i=1}^{m} W_k^{(i)},$ $k = 1, 2, ..., n,$ has the expected value μ_k and variance $\mathrm{var}(\overline{W_k}) = \sigma_k^2 / m.$

The Markov's inequality now states that if Y^* is a positive random variable with expected value $E(Y^*)$, then

$$P(Y^* \geq \alpha E(Y^*)) \leq 1/\alpha, \ \alpha > 1. \tag{2}$$

Inequality (2) implies

$$P(Y^* < \alpha E(Y^*)) \geq 1 - 1/\alpha, \ \alpha > 1. \tag{3}$$

Let us have a nonnegative continuous random variable *Y* defined on a space (Ω, S, P), which has expected value *E*(*Y*). If the variable is changed on the set

$$M = \left\{\omega \in \Omega : \left[Y(\omega) - E(Y)\right] = 0\right\}, \tag{4}$$

i.e. on a set of probability zero since *Y* is a continuous random variable, so that it is strictly positive everywhere in Ω, the change will not alter its stochastic behaviour. Also, the changed variable Y^* will still be a random variable, i.e. a measurable function on *S* (see the short appendix to this paper). Further,

$$0 < E(Y) = E(Y^*), \tag{5}$$

since the two variables differ only on *M*, $P(M) = 0$. Now, applying (2) to Y^*, we

$$P(Y^* \geq \alpha E(Y^*)) \leq 1/\alpha, \ \alpha > 1 \tag{6}$$

and therefore, because of (5) and (6),

$$P(Y \geq \alpha E(Y)) = P(Y \geq \alpha E(Y^*)) \leq P(Y^* \geq \alpha E(Y^*)) \leq 1/\alpha \tag{7}$$

for $\alpha > 1$. Noting the left-hand side and the right-hand side of (7), and applying this result to the nonnegative continuous random variable $\left(\overline{W_k} - \mu_k\right)^2, 1 \leq k \leq n$, we have

$$P\left\{\left(\overline{W_k}-\mu_k\right)^2 \geq \alpha\cdot\operatorname{var}(\overline{W_k})\right\} \leq 1/\alpha \tag{8}$$

for $\alpha > 1, 1 \leq k \leq n,$ as $E\left\{\left(\overline{W_k}-\mu_k\right)^2\right\} = \operatorname{var}(\overline{W_k}).$

Let us rewrite (8) to

$$P\left\{\left|\overline{W_k}-\mu_k\right| \geq \sqrt{\alpha\cdot\operatorname{var}(\overline{W_k})}\right\} \leq 1/\alpha \tag{9}$$

for $\alpha > 1, 1 \leq k \leq n.$ This implies, in the same way as (3) is implied by (2), that

$$P\left\{\left|\overline{W_k}-\mu_k\right| < \sqrt{\alpha\cdot\operatorname{var}(\overline{W_k})}\right\} \geq 1-1/\alpha \quad \text{for } \alpha > 1, 1 \leq k \leq n, \tag{10}$$

or

$$P\left\{\overline{W_k}-\sqrt{\alpha\cdot\operatorname{var}(\overline{W_k})} < \mu_k < \overline{W_k}+\sqrt{\alpha\cdot\operatorname{var}(\overline{W_k})}\right\} \geq 1-1/\alpha \tag{11}$$

for $\alpha > 1, 1 \leq k \leq n.$ To simplify the notation, let the symbol B_k represent the event in the parentheses of (11). We are now interested in the probability $P(\bigcap_{k=1}^n B_k),$ that is in the probability that (11) holds for each k considered at the same time. We would like this probability to be high, but we cannot calculate it as $P(\bigcap_{k=1}^n B_k) = \prod_{k=1}^n P(B_k),$ since the events B_k are not statistically independent. This follows directly from the fact that the sum of $\overline{W_k}\text{'}s$ is one:

$$\sum_{k=1}^n \overline{W_k} = \sum_{k=1}^n (m^{-1}\sum_{i=1}^m W_k^{(i)}) = m^{-1}\sum_i \sum_k W_k^{(i)} =$$
$$= m^{-1}\sum_{i=1}^m 1 = 1. \tag{12}$$

What we can use, however, is the following: First of all, (11) implies

$$\prod_{k=1}^n P(B_k) \geq (1-1/\alpha)^n. \tag{13}$$

Secondly, assuming that among the numbers $P(B_k)$, there is a unique minimum, the relation

$$P(B_h) = \min_{1 \le k \le n} P(B_k) < P(B_l), 1 \le l \le n, l \ne h, \tag{14}$$

implies that

$$B_h \subset B_l, 1 \le l \le n, l \ne h, \tag{15}$$

since the opposite, $B_h \supseteq B_l$ leads to the contradiction

$P(B_h) \ge P(B_l)$. From (15), we obtain

$$B_h = \bigcap_{k=1}^{n} B_k, \tag{16}$$

and so, using (13), (16),

$$\begin{aligned} P(\bigcap_{k=1}^{n} B_k) &= \prod_{k=1}^{n} P(B_k) \cdot \frac{P(\bigcap_{k=1}^{n} B_k)}{\prod_{k=1}^{n} P(B_k)} \\ &\ge \prod_{k=1}^{n} P(B_k) \cdot P(\bigcap_{k=1}^{n} B_k) = \prod_{k=1}^{n} P(B_k) \cdot \min_{1 \le k \le n} P(B_k) \\ &\ge (1 - 1/\alpha)^{n+1}. \end{aligned} \tag{17}$$

To sum up, selecting a high α, we can accomplish that the probability of the event

$$\bigcap_{k} \left\{ \overline{W_k} - \sqrt{\alpha \cdot \operatorname{var}(\overline{W_k})} < \mu_k < \overline{W_k} + \sqrt{\alpha \cdot \operatorname{var}(\overline{W_k})} \right\} \tag{18}$$

is high enough. Therefore, the multivariate interval

$$J = J_1 \times J_2 \times ... \times J_n, \tag{19}$$

where, for $1 \leq k \leq n$,

$$J_k = (J_k^1, J_k^2) = \left(\overline{W_k} - \sqrt{\alpha \cdot \mathrm{var}(\overline{W_k})},\ \overline{W_k} + \sqrt{\alpha \cdot \mathrm{var}(\overline{W_k})}\right) \tag{20}$$

may serve as a confidence interval for the vector of expected or typical weights $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_n)$.

4. PROPER SAMPLE SIZE

In this shorter section of the text, we shall focus on the width of the derived confidence interval. This is important because, as suggested by (17), it might be needed to set α fairly high to achieve an appropriately high probability, especially when n is not particularly small. The problem is that a too large α leads to an overly wide, and therefore imprecise interval (20). This problem brings us to a discussion on how large the random sample we work with should be, since the larger the sample, the smaller the variance $\mathrm{var}(\overline{W_k}) = \sigma_k^2 / m,$ which would allow us a more comfortable setting of α when constructing the interval. Except for the greater precision of the narrower interval, the interval should also be narrow enough not to contain the extreme values zero and one, since if it did contain such values, the result would not comply with our natural and previously mentioned requirement that all conceived product features have their importance in the customers' evaluation of the entire product. To derive how large the random sample should be, we use (17) and (20) with the unknown characteristic $\mathrm{var}(\overline{W_k}) = \sigma_k^2 / m,$ being estimated by its usual sample counterpart $\widehat{\mathrm{var}}(\overline{W_k}) = s_k^2 / m,$ where $s_k^2 = (m-1)^{-1} \sum_{i=1}^{m} (W_k^{(i)} - \overline{W_k})^2, 1 \leq k \leq n.$ As is known, the estimates will be consistent.

The requirement that $(1 - 1/\alpha)^{n+1}$ be sufficiently high, say 0.9, at least, means that

$$\alpha \geq (1 - \sqrt[n+1]{0.9})^{-1} \tag{21}$$

is necessary. The requirement that the intervals don't contain one and zero means that

$$\overline{W_k} - \sqrt{\alpha \cdot \widehat{\mathrm{var}}(\overline{W_k})} > 0$$

and

$$\overline{W_k} + \sqrt{\alpha \cdot \hat{\text{var}}(\overline{W_k})} < 1 \tag{22}$$

for each k considered. Since $\hat{\text{var}}(\overline{W_k})$ depends on the sample size m reciprocally, a large enough sample size will ensure a small enough $\hat{\text{var}}(\overline{W_k})$, which will ensure validity of (22), and, at the same time, will allow for a sufficiently high α that satisfies (21). A four-feature product, for instance, will require α of at least 48 according to (21). If, at the same time, the k-th weight meanders around a value within 0.2 units, and so the variance s_k^2 will be of 0.04 order, then for a $\overline{W_k}$ of 0.2, one can expect a poll of 50 customers to satisfy (22) for this particular k, as well. Thus, setting $\alpha = 48$ and $m = 50$ satisfies (21) and (22) for this k.

5. ADDITIONAL INFORMATION ON $\boldsymbol{\mu}$

Having constructed a confidence interval containing the unknown vector of expected weights $\boldsymbol{\mu}$ with a high probability, we are set to give a thought to where in the interval the unknown vector might be located. Of course, these discussions must be based on additional information. Technically, our discussion is now aimed at selecting a point estimate of $\boldsymbol{\mu}$ from the interval *J*. One obvious and most frequent candidate with this property that would serve as a point estimate of $\boldsymbol{\mu}$ is $(\overline{W_1}, \overline{W_2}, ..., \overline{W_n})$. The coordinates of this vector are consistent estimates of the corresponding coordinates of $\boldsymbol{\mu}$.. The consistency, however, requires larger samples. When a sample is large enough to ensure consistency is not generally known, and a sample of around fifty customers, which otherwise might suffice for other purposes, such as those analyzed in the previous section, is not particularly large by most statistical standards, when it comes to asymptotic statistical properties of an estimate. The question then arises whether a different estimate from *J* than the one typically used could be exploited, provided some other information on the market is available.

Such information may be available indeed, and the kind of information we are going to exploit is the information that a company has an inferior position in the market. This company can be expected to generate a product the value of which is low when judged by customers. Using the weighted-average approach as a model for product value, we could expect that the weighted average of the inferior product is low. We may then be motivated to seek an estimate $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \hat{\mu}_2, ..., \hat{\mu}_n)$ of $\boldsymbol{\mu}$ in *J* which minimizes the weighted-average value of the inferior product. The minimization approximates the low product value. That is, we are looking for such numbers $\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \hat{\mu}_2, ..., \hat{\mu}_n),$ that for the following function $f(\mathbf{w}) = f(w_1, ..., w_n) = \sum_{i=1}^{n} h_i w_i$ and the set $M = \left\{\mathbf{w} \in R^n : J_i^1 \le w_i \le J_i^2, i = 1, 2, ..., n, \sum_{i=1}^{n} w_i = 1\right\},$ we have

$$\min_{\mathbf{w} \in M} f(\mathbf{w}) = f(\hat{\boldsymbol{\mu}}). \tag{23}$$

The coefficients h_i's in f are the levels of the features of an inferior product. Three questions are of interest in connection with (23): A) Is there a point $\hat{\boldsymbol{\mu}} \in R^n$ which minimizes f on M? B) If such a point exists, is it the only solution to the problem? C) How to find the point $\hat{\boldsymbol{\mu}}$ if (23) holds?

The answer to question A) is as follows: firstly, $M \neq \emptyset$ according to (11) and (12); secondly, the set is apparently bounded; thirdly, the set is closed – a point $\mathbf{z}$ with a zero distance from the set $K = \left\{\mathbf{w} \in R^n : \sum_{i=1}^{n} w_i = 1\right\}$ is the limit of a sequence of points $\mathbf{x}^{(m)}$ from this set. Now, since the function $k(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} x_i - 1$ is continuous, $k(\mathbf{z}) = \lim_{m \to +\infty} k(\mathbf{x}^{(m)}) = \lim_{m \to +\infty} 0 = 0,$ that is $\sum_{i=1}^{n} z_i - 1 = 0$ and $\mathbf{z} \in K.$ The set K is therefore closed. Since M is an intersection of two closed sets – the interval J and the set K, it is closed itself. Further, the function f is continuous on M, and so the existence of a solution to the problem (23) is guaranteed.

Let us now focus on question C). Although a solution may easily be found with an optimization software, since we are dealing with a linear program, we shall first discuss a way of detecting the solution without software because the discussion will also help us answer question B). Later, however, we will use a simple software package to verify the validity of our conclusions.

Since the solution is an element of $J \cap K$, it belongs to J. Let us imagine that the solution $\hat{\boldsymbol{\mu}}$ exists in the interior of J. Since there are integers $k, l,$ $1 \le k \le n, 1 \le l \le n, k \neq l,$ such that $h_k < h_l$ (see the assumptions on the h_l's), selecting a vector $\mathbf{u} \in R^n$, $\mathbf{u} = (0, 0, ..., \varepsilon, 0, ..., -\varepsilon, 0, ..., 0),$ where ε is positive and represents the k-th coordinate of $\mathbf{u}$ and $-\varepsilon$ represents the l-th coordinate of the vector, we get

$$\varepsilon h_k < \varepsilon h_l, \text{ or } \varepsilon h_k - \varepsilon h_l < 0 \tag{24}$$

and

$$f(\hat{\boldsymbol{\mu}}+\mathbf{u}) = f(\hat{\boldsymbol{\mu}}) + f(\mathbf{u}) = f(\hat{\boldsymbol{\mu}}) + \varepsilon h_k - \varepsilon h_l < f(\hat{\boldsymbol{\mu}}). \tag{25}$$

At the same time, for a small enough positive ε, the point $\hat{\boldsymbol{\mu}}+\mathbf{u} = (\hat{\mu}_1 + u_1, ..., \hat{\mu}_n + u_n)$, resulting from shifting the point $\hat{\boldsymbol{\mu}}$ in the direction of $\mathbf{u}$ by a unit, will still belong to *J*, since $\hat{\boldsymbol{\mu}}$ is an interior point of the interval. Moreover, $\sum_{i=1}^{n}(\hat{\mu}_i + u_i) = 1$ and $\hat{\boldsymbol{\mu}}+\mathbf{u}$ belongs to *K*, as well. Thus, our prerequisite of having a solution in the interior of *J* leads to a contradiction – we have found another point in *M* which reduces the already minimal value of the function. The current conclusion is the solution lies on the boundary of *J*, in other words, at least one of the coordinates of the solution, say $\hat{\mu}_i$, for instance, lies on the boundary of the corresponding univariate interval J_i.

Remark: If $M^* = \left\{\mathbf{w} \in R^n : J_i^1 \le w_i \le J_i^2, i = 1, ..., n, \sum_{i=1}^{n} w_i = c\right\}$, $c \in (0, 1)$ fixed, is defined instead of *M* and $M^* \ne \varnothing$ holding, then, replacing $\sum_{i=1}^{n}(\hat{\mu}_i + u_i) = 1$ with $\sum_{i=1}^{n}(\hat{\mu}_i + u_i) = c$ in the proof and keeping the rest of the problem the same, the conclusions about the existence and boundary location of the solution for the newly defined set M^*, or the newly defined problem, will apply, as well.

Let us return to the original problem now. The coordinate of the solution which lies on the boundary of its corresponding interval is one of the numbers $\hat{\mu}_i$, $1 \le i \le n.$ Let $\hat{\mu}_1$ be such a number. Since $\hat{\mu}_1 \in J_1$, $0 < \hat{\mu}_1 < 1$ (see section 4). The other coordinates of the solution, the numbers $\hat{\mu}_2, ..., \hat{\mu}_n$, satisfy

$$\hat{\mu}_i \in J_i, i = 2, ..., n, \sum_{i=2}^{n} \hat{\mu}_i = 1 - \hat{\mu}_1. \tag{26}$$

The numbers also minimize the function

$$g(w_2, w_3, ..., w_n) = h_1\hat{\mu}_1 + \sum_{i=2}^{n} h_i w_i \tag{27}$$

on the set

$$M^{*}=\left\{\mathbf{w}:J_i^1\le w_i\le J_i^2,\, i=2,...,n,\, \sum_{i=2}^{n} w_i=1-\hat{\mu}_1\right\}. \tag{28}$$

Thus, to find the other coordinates $\hat{\mu}_2,...,\hat{\mu}_n$, we need to solve the subproblem (26) – (28), but this is exactly the kind of problem discussed in the remark, provided the h_i's in (27) are mutually different, which is the case here by our assumptions. Strictly speaking, the term $h_1\hat{\mu}_1$ should also be included in (25), but this does not alter the conclusion: $g(\hat{\boldsymbol{\mu}}+\mathbf{u})=g(\hat{\boldsymbol{\mu}})+g(\mathbf{u})-h_1\hat{\mu}_1=g(\hat{\boldsymbol{\mu}})+\varepsilon h_k-\varepsilon h_l<g(\hat{\boldsymbol{\mu}})$. Thus, the subproblem has a solution on the boundary of $J_2\times...\times J_n$, i.e. at least one of the coordinates $\hat{\mu}_2,...,\hat{\mu}_n$ is on the boundary of its corresponding univariate interval. Let us say that it is $\hat{\mu}_2$. Then we may proceed in the same way as we have done so far, encountering another sub-subproblem with a solution which is on the boundary of the interval $J_3\times...\times J_n$. Proceeding this way, each of the numbers $\hat{\mu}_1,...,\hat{\mu}_n$, except perhaps one, $\hat{\mu}_{j_n}$, which will be calculated in the final iteration of the procedure as a straightforward solution of the form $\hat{\mu}_{j_n}=1-\sum_{k=1}^{n-1}\hat{\mu}_{j_k}$, will lie on the boundary of its corresponding univariate interval.

We don't know which sequence $\hat{\mu}_{j_1}\to\hat{\mu}_{j_2}\to...\to\hat{\mu}_{j_n}$ we should decide to pass through, and which of the two boundaries of the interval $J_{j_i}, 1\le i\le n$, to insert in $\hat{\mu}_{j_i}, 1\le i\le n$, on this path, so we might try all the combinations. One of the combinations will lead to the solution. This procedure leads to setting up a tree which, however, need not be walked through entirely, as suggested, since many of the paths repeat. Since the tree has at most a finite number of branches, the number of solutions to the original problem is finite. We shall see, however, that there is only one solution to the problem. To see this, we shall draw on the conclusion of our discussion so far that at most one of the weights representing the solution does not lie on the boundary of its corresponding univariate interval. This said, let us look at question B).

Let's assume more than one solution to the problem exist, i.e. there is a point $\mathbf{P}=(w_1^p, w_2^p, ..., w_n^p)$ and a point $\mathbf{Q}=(w_1^q, w_2^q, ..., w_n^q)$, $\mathbf{P}\ne\mathbf{Q}$, both of which minimize the function f on M, that is $f(\mathbf{P})=f(\mathbf{Q})$. An index i, $1\le i\le n$, exists, such that $w_i^p\ne w_i^q$. Since both **P** and **Q** represent a set of weights, we have

$$w_1^p+w_2^p+...+w_{i-1}^p+w_{i+1}^p+...+w_n^p=1-w_i^p, \tag{29}$$

$$w_1^q+w_2^q+...+w_{i-1}^q+w_{i+1}^q+...+w_n^q=1-w_i^q. \tag{30}$$

Since the right-hand sides of (29) and (30) differ, another index $j, 1 \le j \le n,\ j \ne i,$ exists that $w_j^p \ne w_j^q$. Since J is convex, the line $\mathbf{P} + \alpha(\mathbf{Q} - \mathbf{P}),\ \alpha \in [0, 1],$ belongs to J, moreover, $\sum_{k=1}^{n} \left[w_k^p + \alpha(w_k^q - w_k^p) \right] = 1,\quad f(\mathbf{P} + \alpha(\mathbf{Q} - \mathbf{P})) = f(\mathbf{P}) + \alpha(f(\mathbf{Q}) - f(\mathbf{P})) = f(\mathbf{P}),$ so all the points on the line segment are solutions to the problem. Since **P** is a solution, either w_i^p or w_j^p (or both) lies on the boundary of the interval J_i and J_j respectively. If none of the numbers lied on the boundary, P would not be a solution, as we saw in the previous discussion. However, looking at a point $\mathbf{P} + \alpha(\mathbf{Q} - \mathbf{P}),\ \alpha \in (0, 1),$ and its *i*-th and *j*-th coordinate, in particular,

$$w_i^p + \alpha(w_i^q - w_i^p), \tag{31}$$

$$w_j^p + \alpha(w_j^q - w_j^p), \tag{32}$$

we see that the coordinates $w_i^p,\ w_j^p$ will move away from their respective boundaries if they are on the boundaries, or one of them will stay away from its boundary if it isn't already on its boundary, for a sufficiently small $\alpha \in (0, 1),$ since $w_i^p \ne w_i^q,\ w_j^p \ne w_j^q.$ Yet, the moved point will remain, as we saw in the previous paragraph, a solution to the problem. This solution will have at least two coordinates not lying on the boundaries of their respective intervals. This, however, is a contradiction. Thus, there cannot be two different solutions to the problem.

As was mentioned at the beginning of the analysis, the unique solution can be found with a software. We went deeper into the problem to see that results returned by an optimization software may be considered unique. We shall use now, as promised, a simple software - the Excel Solver to demonstrate the validity of the presented ideas.

6. EXAMPLE

The final section presents an example to demonstrate the procedure just described, and verify its validity, using Excel Solver. A four-feature product is analysed. The value function is $f(w_1, w_2, w_3, w_4) = 0.11w_1 + 0.4w_2 + 0.2w_3 + 0.7w_4$, and let us assume that the confidence interval for the weights is $J = [0.2, 0.5] \times [0.05, 0.3] \times [0.1, 0.2] \times [0.3, 0.35].$

The aforementioned tree has 128 branches, representing all possible sequences of selecting the boundary values for the unknown expected weights. The sequences are:

$$*w_1 \to w_2 \to w_3 \to w_4,\ *w_1 \to w_2 \to w_4 \to w_3,\ w_1 \to w_3 \to w_2 \to w_4,\ *w_1 \to w_3 \to w_4 \to w_2,$$
$$w_2 \to w_1 \to w_3 \to w_4,\ w_2 \to w_1 \to w_4 \to w_3,\ w_2 \to w_3 \to w_1 \to w_4,\ *w_2 \to w_3 \to w_4 \to w_1,$$
$$w_3 \to w_1 \to w_2 \to w_4,\ w_3 \to w_1 \to w_4 \to w_2,\ w_3 \to w_2 \to w_1 \to w_4,\ w_3 \to w_2 \to w_4 \to w_1,$$
$$w_4 \to w_1 \to w_2 \to w_3,\ w_4 \to w_1 \to w_3 \to w_2,\ w_4 \to w_2 \to w_1 \to w_3,\ w_4 \to w_2 \to w_3 \to w_1.$$

In each of the w_i's in each sequence, one of the two boundaries of the corresponding univariate interval may be inserted. For instance, taking the first sequence, we may insert 0.2 or 0.5 in w_1, and for any of the two values, we may then insert 0.05 or 0.3 in w_2, and for each of the four scenarios, we can finally insert 0.1 or 0.2 in w_3. The last weight is $w_4 = 1 - w_1 - w_2 - w_3$. Working with each of the sequences this way provides $16 \cdot 8 = 128$ scenarios. Each time a set of weights is selected, the function f is evaluated for the set, and the value which minimizes f determines the weights from J we looked for. We don‘t need to evaluate all 128 sets of weights, however, but only the ones marked with "*", since the other sets are a repetition. These are sequences 1, 2, 4 and 8. For the sequence 1, the possible weights are

Table 1 Sets of weights for the first sequence

w1	**w2**	**w3**	**w4**	**f**
0.2	0.05	0.1	0.65	
0.2	0.05	0.2	0.55	
0.2	0.3	0.1	0.4	
0.2	0.3	0.2	0.3	0.39
0.5	0.05	0.1	0.35	0.34
0.5	0.05	0.2	0.25	
0.5	0.3	0.1	0.1	
0.5	0.3	0.2	0	

Only the highlighted sets are of interest to us, since the other fall out of J, and therefore cannot be the solution from M. The column „f“ records the value of the function f for the corresponding set of weights. Similarly, for the sequence 2,

Table 2 Sets of weights for the second sequence

w1	w2	w4	w3	f
0.2	0.05	0.3	0.45	
0.2	0.05	0.35	0.4	
0.2	0.3	0.3	0.2	0.392
0.2	0.3	0.35	0.15	0.417
0.5	0.05	0.3	0.15	0.315
0.5	0.05	0.35	0.1	0.34
0.5	0.3	0.3	-0.1	
0.5	0.3	0.35	-0.15	

Again, we pay attention only to the highlighted sets. For the sequence 4, we have

Table 3 Sets of weights for the fourth sequence

w1	w3	w4	w2	f
0.2	0.1	0.3	0.4	
0.2	0.1	0.35	0.35	
0.2	0.2	0.3	0.3	0.392
0.2	0.2	0.35	0.25	0.407
0.5	0.1	0.3	0.1	0.325
0.5	0.1	0.35	0.05	0.34
0.5	0.2	0.3	0	
0.5	0.2	0.35	-0.05	

and for the final sequence 8,

Table 4 Sets of weights for the eighth sequence

w2	w3	w4	w1	f
0.05	0.1	0.3	0.55	
0.05	0.1	0.35	0.5	0.34
0.05	0.2	0.3	0.45	0.3195
0.05	0.2	0.35	0.4	0.349
0.3	0.1	0.3	0.3	0.383
0.3	0.1	0.35	0.25	0.4125
0.3	0.2	0.3	0.2	0.392
0.3	0.2	0.35	0.15	

Comparing the values of f for the feasible solutions, the sought unique minimum of the function is attained at $\left[0.5, 0.05, 0.15, 0.3\right]$. The same result is achieved with Excel when the same problem: Minimize $f(w_1, w_2, w_3, w_4) = 0.11w_1 + 0.4w_2 + 0.2w_3 + 0.7w_4$, on the set $\left[0.2, 0.5\right]\times\left[0.05, 0.3\right]\times\left[0.1, 0.2\right]\times\left[0.3, 0.35\right]\cap\left\{\mathbf{w} : \sum w_i = 1\right\}$ is resolved.

7. CONCLUSIONS

In this paper, a procedure was presented that estimates how a selected market segment characterized by products of the same type perceives the products. A weighted-average approach was used as a model for the product perception or evaluation. Since this approach requires weights to be assigned to product features by the market segment, and the weights are usually unknown and vary from customer to customer, the procedure focused on estimating „average“ or expected weights of the entire market segment, which may be regarded as representative or prevailing weights on the market. The estimation was performed by constructing a confidence interval for the expected weights, the interval being derived generally on the basis of a random poll run across a sufficiently large sample of customers, and having the property of containing the unknown expected weights with a predefined high probability. Further, a more precise location of the expected weights in the interval was suggested in the paper, the intended accuracy being based on an additional piece of information regarding inferior product makers. This information suggests that if a product maker is viewed as inferior, its product value must be low. Using the weighted-average approach as a model for product evaluation, a set of expected weights from the interval which minimizes the value of the inferior product was proposed as the appropriate one, the minimization serving as an approximation of what „low value“ means. Since a minimization was involved in the problem, existence and uniqueness, in particular, of the solution to the problem was discussed and confirmed.

APPENDIX

When constructing a confidence interval for the vector of unknown expected weights, we changed a nonnegative continuous random variable *Y*, defined on Ω of a probability space (Ω, S, P), so that it was strictly positive everywhere in Ω. This can be done by changing *Y* on the set $N = \{\omega \in \Omega : Y(\omega) = 0\} \in S$ to a positive constant *c*, and keeping the variable the same everywhere else, i.e. on the set $\Omega - N$. The changed variable Y^* will still be a random variable or a measurable function, that is, if *r* is any real number, then $K = \{\omega \in \Omega : Y^*(\omega) < r\} \in S$. To see this, let us select a real number *r*. If $r \le 0$, then $K = \varnothing$, which always belongs to the $\sigma-$algebra *S* of events. If $r > 0$, then either $r > c$, or $r \le c$. Since *Y* is a measurable function on *S*, we have: for the case $r > c$, $K = \{\omega \in \Omega : Y^*(\omega) < r\} = \{\omega \in \Omega : Y(\omega) < r\} \in S$; if $r \le c$, $K = \{\omega \in \Omega : Y^*(\omega) < r\} = \{\omega \in \Omega : Y(\omega) < r\} - N \in S$.

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