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DISTANCE MEASURES FOR HESITANT INTUITIONISTIC FUZZY LINGUISTIC SETS

Abstract. *To address qualitative and quantitative preferences, this paper defines a new fuzzy set named hesitant intuitionistic fuzzy linguistic sets (HIFLSs), which are composed by several linguistic terms with each of them having an intuitionistic fuzzy set. To research the application of HIFLSs, several distance measures are defined. To cope with the situations where the elements in a set are correlative, three interrelated distance measures are presented. To address intuitionistic hesitant fuzzy linguistic decision making problems with incomplete weight information, models for the optimal 2-additive measures and additive measures are respectively built. Then, an approach to multi-attribute decision making is developed. Finally, an illustrative example is offered to demonstrate the application of the procedure.*

Key words: *decision making, hesitant intuitionistic fuzzy linguistic set, distance measure, 2-additive measure.*

JEL Classification: D81, M12, M51

1. Introduction

With the socioeconomic development, decision-making problems become more and more complex, and there are usually many uncertain factors. As is in the real life, it is usually difficult to require decision makers to denote the attribute preferences of alternatives by using one crisp or fuzzy number. To cope with this issue, researchers developed multi-attribute decision making under hesitant fuzzy environment. Torra (2010) first noted this problem and introduced the concept of hesitant fuzzy sets (HFSs), which permit the decision maker to apply several values to express his/her preference. After the pioneer work of Torra (2010), HFSs are studied by many researchers (Chen et al., 2013; Xia and Xu, 2011; Xu and Xia, 2011; Meng et al., 2016).

Linguistic variables are an effective tool to express the decision makers' qualitative preferences. Since Zadeh (1975) first introduced the concept of linguistic variables, some generalized forms are developed, such as interval linguistic variables (Xu, 2004) and hesitant fuzzy linguistic term sets (HFLTSs) (Rodríguez et al., 2012). To research the application of linguistic variables, many computing with words (CW) models are presented (Yager, 2004; Türkşen, 2002; Herrera and Herrera-Viedma, 2000; Herrera and Martínez, 2000; Zhang, 2012).

Although there are many researches about linguistic variables, all above mentioned references only reflect the qualitative preferences of the decision makers. To address the quantitative preferences of the decision makers with respect to linguistic variables, Isitt (1990) introduced proportional linguistic term sets that are expressed by several linguistic terms with the associated percentage values. For example, to evaluate the quietness of an engine, an decision maker may state that he/she is 30% sure it is slightly good, 40% sure it is good, and 30% sure it is very good, expressed by S(quietness)={(slightly good, 0.3), (good, 0.4), (very good, 0.3)}. However, proportional linguistic term sets only consider the sure percentage and do not give the false and uncertain percentages, while intuitionistic linguistic sets only allow the decision maker to express their qualitative preferences using one linguistic variable. As Torra (2010) and Rodríguez et al. (2012) noted, in some situations, it may be more suitable to express the decision maker's preferences using several possible values. Considering this issue, Meng et al. (2014) introduced linguistic hesitant fuzzy sets to express the decision maker's qualitative and quantitative preferences. However, this kind of fuzzy sets can only denote the membership degree of the attribute preferences of alternatives, but the non-membership degree does not consider.

Based on above analyses, this paper presents a new kind of fuzzy sets called hesitant intuitionistic fuzzy linguistic sets (HIFLSs), which permit the decision makers to express their qualitative preferences using several linguistic terms with each of them having an intuitionistic fuzzy set to denote their quantitative fuzzy preferences. This kind of fuzzy sets can be seen as an extension of several kinds of fuzzy sets, such as proportional linguistic term sets (Isitt, 1990), intuitionistic linguistic sets (Liu and Jin, 2012) and hesitant fuzzy linguistic term sets (Rodríguez et al., 2012). HIFLSs not only denote the decision maker's quantitative preferences but also express his/her qualitative preferences. Meanwhile, HIFLSs both reflect the membership and non-membership degrees of the decision maker's quantitative preferences. For example, to evaluate the picture quality of a TV, an decision maker may state that he/she is 20% sure it is "fair" and 60% sure it is not "fair", 50% sure it is "clear" and 30% sure it is not "clear", 30% sure it is "very clear" and 40% sure it is not "very clear". To express this preference, HIFLSs is a good choice.

The rest part of this paper is organized as follows: Section 2 first reviews some basic concepts and then introduces the concept of HIFLSs. Meanwhile, a distance measure is defined. Section 3 defines two kinds of intuitionistic hesitant fuzzy linguistic weighted distance measures. One is based on the assumption that the elements in a set are independent; the other applies 2-addtive measures to address their interactions. Section 4 introduces a correlation coefficient of HIFLSs, by which models for the optimal weight vector are constructed. Then, a decision method for intuitionistic hesitant fuzzy linguistic decision making is developed. Section 5 offers a practical example to demonstrate the practicality of the procedure. The conclusion is shown in the last section.

2. The concept of hesitant intuitionistic fuzzy linguistic sets

As we know, it is an effective tool to use linguistic variables to cope with decision-making problems with qualitative attributes. For example, to evaluate the morality of a person, it can use the linguistic variable "bad", "fair", or "good". To further consider the application of linguistic variables, it is usually assumed that linguistic variables are defined on an linguistic term set with odd cardinality, denoted $S = \{s_i \mid i = 0, 1, \ldots, t\}$, where s_i represents a possible value for a linguistic variable. With respect to the linguistic term set $S = \{s_i \mid i = 0, 1, \ldots, t\}$, Herrera and Martinez (2000) introduced the following characteristics: (i) The set is ordered: $s_i > s_j$, if $i > j$; (ii) Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$; (iii) Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$; (iv) A negation operator: $\mathrm{neg}(s_i) = s_j$ such that $j = t-i$.

To deal with uncertainty and hesitancy of the decision makers' qualitative references, several extended forms are developed (Xu, 2004; Rodríguez et al., 2012; Isitt, 1990; Meng et al., 2014). From the example given in introduction, we know that no kinds of linguistic variables could express the decision maker's preference for evaluating the picture quality of a TV.

Now, let us introduce a new linguistic set called hesitant intuitionistic fuzzy linguistic sets.

Definition 1. Let $S=\{s_1, \ldots, s_t\}$ be a linguistic term set. An HIFLS in S is a set that when applied to the linguistic terms of S, returns a subset of S, denoted $IH=\{<s_{\theta(i)}, (\alpha_{(i)}, \beta_{(i)})> | s_{\theta(i)} \in S)\}$, where $\alpha_{(i)}$ and $\beta_{(i)}$ are values in [0, 1] denoting the membership and non-membership degrees of the linguistic variable $s_{\theta(i)}$ to IH with $\alpha_{(i)}+\beta_{(i)} \leq 1$.

Next, Let us consider a distance measure if HIFLSs:

Definition 2. Let $S=\{s_1, \ldots, s_t\}$ be the predefined linguistic term set. The distance measure**,** for any two HIFLSs $IH_1=\{<s_{\theta(i)}, (\alpha_{(i)}, \beta_{(i)})> | s_{\theta(i)} \in S)\}$and $IH_2=\{<s_{\theta(j)}, (\alpha_{(j)}, \beta_{(j)})> | s_{\theta(j)} \in S)\}$, is defined by

$$d(IH_1, IH_2) = \frac{d_{IH_1 \to IH_2} + d_{IH_2 \to IH_1}}{2}, \tag{1}$$

where

$$d_{IH_1 \to IH_2} = \frac{1}{c(IH_1)} \sum_{\langle s_{\theta(i)}, (\alpha_{(i)}, \beta_{(i)}) \rangle \in IH_1} \min_{\langle s_{\theta(j)}, (\alpha_{(j)}, \beta_{(j)}) \rangle \in IH_2} \frac{|\theta(i)\alpha_{(i)} - \theta(j)\alpha_{(j)}| + |\theta(i)\beta_{(i)} - \theta(j)\beta_{(j)}|}{2t},$$

$$d_{IH_2 \to IH_1} = \frac{1}{c(IH_2)} \sum_{\langle s_{\theta(j)}, (\alpha_{(j)}, \beta_{(j)}) \rangle \in IH_2} \min_{\langle s_{\theta(i)}, (\alpha_{(i)}, \beta_{(i)}) \rangle \in IH_1} \frac{|\theta(j)\alpha_{(j)} - \theta(i)\alpha_{(i)}| + |\theta(j)\beta_{(j)} - \theta(i)\beta_{(i)}|}{2t}$$

with $c(IH_1)$ and $c(IH_2)$ being the numbers of linguistic terms in IH_1 and IH_2, respectively.

Proposition 1. Let IH_1 and IH_2 be any two HIFLSs for the predefined linguistic term set $S=\{s_1, \ldots, s_t\}$. Then, the distance measure d has the following characteristics:

(i) $d(IH_1, IH_1) = 0$;
(ii) $0 \le d(IH_1, IH_2) \le 1$;
(iii) $d(IH_1, IH_2) = d(IH_2, IH_1)$.

3. The distance measure

The distance measure is an effective tool to obtain the comprehensive attribute values of alternatives, which is widely used in decision-making problems (Zeng and Su, 2011; Xu and Xia, 2011; Peng et al., 2013, Xu, 2012).

Based on the distance measure given in Definition 2, the section dedicates to introduce several distance measures on HIFLSs

3.1. The distance measure based on additive measures

Definition 3. Let $X=\{IH_1, IH_2, ..., IH_n\}$ and $Y=\{IH'_1, IH'_2, ..., IH'_n\}$ be any two sets of HIFLSs for the predefined linguistic term set $S=\{s_1, \ldots, s_t\}$. The generalized intuitionistic hesitant fuzzy linguistic weighted distance (GWD^{HIFL}) measure between X and Y is defined by

$$\text{GWD}_{\omega}^{\text{IHFL}}(X,Y)=\left(\sum_{i=1}^{n}\omega_i d(IH_i, IH'_i)^r\right)^{\frac{1}{r}},$$

where $r \in \text{R}^+$, and ω_i is the weight of $d(IH_i, IH'_i)$ with $\sum_{i=1}^{n}\omega_i = 1$ and $\omega_i \ge 0$.

Definition 4. Let $X=\{IH_1, IH_2, ..., IH_n\}$ and $Y=\{IH'_1, IH'_2, ..., IH'_n\}$ be any two sets of HIFLSs for the predefined linguistic term set $S=\{s_1, \ldots, s_t\}$. The generalized intuitionistic hesitant fuzzy linguistic ordered weighted distance ($\text{GOWD}^{\text{HIFL}}$) measure between X and Y is defined by

$$\text{GOWD}_{w}^{\text{IHFL}}(X,Y)=\left(\sum_{i=1}^{n}w_i d(IH_{(i)}, IH'_{(i)})^r\right)^{\frac{1}{r}},$$

where $r \in \text{R}^+$, $(\cdot)$ is a permutation on $N=\{1, 2, \ldots, n\}$ such that $d(IH_{(i)}, IH'_{(i)})$ is the ith least value of $d(IH_i, IH'_i)$, $i \in N$, and w_i is the weight of the ith position with

$\sum_{i=1}^{n} w_i = 1$ and $w_i \geq 0$.

The GWD$^{\mathrm{HIFL}}$ measure only gives the importance of the elements and the GOWD$^{\mathrm{HIFL}}$ measure only considers the importance of the ordered positions. To consider these two aspects, let us define the hybrid distance measure as follows:

Definition 5. Let $X = \{IH_1, IH_2, ..., IH_n\}$ and $Y = \{IH'_1, IH'_2, ..., IH'_n\}$ be any two sets of HIFLSs for the predefined linguistic term set $S=\{s_1, \ldots, s_t\}$. The generalized intuitionistic hesitant fuzzy linguistic hybrid weighted distance (GHWD$^{\mathrm{HIFL}}$) measure between X and Y is defined by

$$\mathrm{GOWD}_{\omega,w}^{\mathrm{IHFL}}(X,Y) = \left(\sum_{i=1}^{n} \frac{w_i \omega_{(i)} d(IH_{(i)}, IH'_{(i)})^r}{\sum_{i=1}^{n} w_i \omega_{(i)}} \right)^{\frac{1}{r}},$$

where $r \in \mathrm{R}^+$, $(\cdot)$ is a permutation on $N=\{1, 2, \ldots, n\}$ such that $\omega_{(i)} d(IH_{(i)}, IH'_{(i)})$ is the ith least value of $\omega_i d(IH_i, IH'_i)$, $i \in N$, ω_i is the weight of $d(IH_i, IH'_i)$ with $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$, and w_i is the weight of the ith position with $\sum_{i=1}^{n} w_i = 1$ and $w_i \geq 0$.

3.2. The distance measure based on 2-addtive measures

The distance measures given in subsection 3.1 are all based on additive measures (or probability measures) that can only deal with the situation where elements in a set are independent. However, in general, there are some degrees of interactions between elements (Grabisch, 1995). To cope with this situation, researchers usually adopt fuzzy measures (Sugeno, 1974): A fuzzy measure μ on $X=\{x_1, x_2, \ldots, x_n\}$ is a set function $\mu : P(X) \to [0,1]$ satisfying (i) $\mu(\varnothing) = 0$, $\mu(X) = 1$; (ii) $A, B \in P(X)$ such that $A \subseteq B$ implies $\mu(A) \leq \mu(B)$, where $P(X)$ denotes the power set of X. From the concept of fuzzy measures, we know that the fuzzy measure is defined on the power set. This makes the problem exponentially complex.

To simplify the complexity of solving a fuzzy measure, this section develops several distance measures based on 2-additve measures.

Definition 6 (Grabisch, 1997)**.** A fuzzy measure μ on N={1, 2,…, n} is said to be a 2-additive measure, if, for any $S \subseteq N$ with $s \geq 2$, we have

$$\mu(S) = \sum_{\{i,j\}\subseteq S} \mu(i,j) - (s-2)\sum_{i\in S}\mu(i), \tag{2}$$

where s is the cardinality of S.

In a multi-attribute decision making problem, when there are interactions between attributes, then how to determine their weights? To answer this question, the Shapley function (Shapley, 1953) may be a good choice, denoted by

$$Sh_i(\mu,N) = \sum_{T\subseteq N\backslash i} \frac{(n-t-1)!t!}{n!}(\mu(i\cup T)-\mu(T)) \quad \forall i \in N, \tag{3}$$

where μ is a fuzzy measure, n and t are the cardinalities of N and T, respectively.

Especially, when μ is a 2-additive measure, then it derives (Meng and Tang, 2013):

$$Sh_i(\mu,N) = \frac{3-n}{2}\mu(i) + \frac{1}{2}\sum_{j\in N\backslash i}\left(\mu(i,j)-\mu(j)\right) \qquad \forall i \in N. \tag{4}$$

From monotonicity of fuzzy measure, one can easily derive that $\{Sh_i(\mu, N)\}_{i\in N}$ is a weighting vector. Now, let us define the distance measures based on 2-additive measures.

Definition 7. Let $X = \{IH_1, IH_2, ..., IH_n\}$ and $Y = \{IH'_1, IH'_2, ..., IH'_n\}$ be any two sets of HIFLSs for the predefined linguistic term set S={s_1, …, s_t}. The generalized intuitionistic hesitant fuzzy linguistic 2-additive Shapley weighted distance (G2SWD$^{\text{HIFL}}$) measure between X and Y is defined by

$$\text{G2SWD}_{\mu}^{\text{IHFL}}(X,Y) = \left(\sum_{i=1}^{n} Sh_i(\mu, D) d(IH_i, IH'_i)^r\right)^{\frac{1}{r}},$$

where $r\in\mathrm{R}^+$, and $Sh_i(\mu, D)$ is the Shapley value of $d(IH_i, IH'_i)$ with μ being a 2-additive measure on D={ $d(IH_i, IH'_i)$ }$_{i=1,2,\ldots,n}$.

Corresponding to fuzzy measures, fuzzy integrals are an effective tool to aggregate information with interactive characteristics. In 1995, Grabisch (1995) gave the explicit expression of the Choquet integral on finite discrete sets.

Definition 8 (Grabisch, 1995)**.** Let f be a positive real-valued function on $X=\{x_1,x_2,...,x_n\}$, and μ be a fuzzy measure on X. The discrete Choquet integral of f for μ is defined by

$$C_\mu(f(x_{(1)}), f(x_{(2)}),..., f(x_{(n)}))=\sum_{i=1}^{n} f(x_{(i)})(\mu(A_{(i)})-\mu(A_{(i+1)})), \tag{5}$$

where $(\cdot)$ indicates a permutation on X such that $f(x_{(1)})\le f(x_{(2)})\le\ldots\le f(x_{(n)})$, and $A_{(i)}=\{x_{(i)},...,x_{(n)}\}$ with $A_{(n+1)}=\varnothing$.

In Definition 8, if μ is a 2-additive measure, then Eq.(5) can be equivalently expressed by

$$C_\mu(f(x_{(1)}), f(x_{(2)}),..., f(x_{(n)}))=\sum_{i=1}^{n} f(x_{(i)})\left(\sum\nolimits_{j=i+1}^{n}\left(\mu(x_{(i)},x_{(j)})-\mu(x_{(j)})\right)-(n-i-1)\mu(x_{(i)})\right).$$

The Choquet integral addresses the importance of the ordered positions by considering the adjacent coalitions. However, when there are interactions, it seems to be unreasonable. Here, we use the generalized Shapley function (Marichal, 2000) to denote the importance of the ordered coalitions, denoted by

$$GSh_S(\mu,N)=\sum_{T\subseteq N\setminus S}\frac{(n-t-s)!t!}{(n-s+1)!}(\mu(S\cup T)-\mu(T)) \qquad \forall S\subseteq N, \tag{6}$$

where μ is a fuzzy measure, n, t and s denote the cardinalities of N, T and S, respectively.

In Eq.(6), when μ is a 2-additive measure, Meng and Tang (2013) gave the following conclusion.

Theorem 1 (Meng and Tang, 2013)**.** Let μ be a 2-fuzzy measure defined on N={1, 2, …, n}, and *GSh* be the generalized Shapley function given in Eq.(6). Then,

$$GSH_{S\cup\{i\}}(\mu,N)-GSH_{S}(\mu,N)=Sh_{i}(\mu,N) \tag{7}$$

for any $i \in N$ and any $S \subseteq N$ with $i \notin S$.

Definition 9. Let f be a positive real-valued function on $X=\{x_1,x_2,...,x_n\}$, and μ be a 2-additive measure on X. The discrete Choquet integral of f for *GSh* is defined by

$$C_{\mu}(f(x_{(1)}),f(x_{(2)}),...,f(x_{(n)}))=\sum_{i=1}^{n} f(x_{(i)})Sh_{(i)}(\mu,N), \tag{8}$$

where (·) indicates a permutation on X such that $f(x_{(1)})\le f(x_{(2)})\le\ldots\le f(x_{(n)})$, and $A_{(i)}=\{x_{(i)},...,x_{(n)}\}$ with $A_{(n+1)}=\varnothing$.

Similar to the GWD$^{\text{HIFL}}$ measure, the G2SWD$^{\text{HIFL}}$ measure only gives the importance of elements, and does not consider the importance of the ordered positions. Next, let us give another distance measure based on 2-additive measures.

Definition 10. Let $X=\{IH_1,IH_2,...,IH_n\}$ and $Y=\{IH'_1,IH'_2,...,IH'_n\}$ be any two sets of HIFLSs for the predefined linguistic term set S={s_1, …, s_t}. The generalized intuitionistic hesitant fuzzy linguistic Choquet 2-additive Shapley weighted distance (GC2SWD$^{\text{HIFL}}$) measure between X and Y is defined by

$$\text{GC2SWD}_{v}^{\text{IHFL}}(X,Y)=\left(\sum_{i=1}^{n} Sh_i(v,N)d(IH_{(i)},IH'_{(i)})^r\right)^{\frac{1}{r}},$$

where $r\in$R\0, (·) is a permutation on N={1, 2, …, n} such that $d(IH_{(i)},IH'_{(i)})$ is the ith least value of $d(IH_i,IH'_i)$, $i\in N$, and $Sh_i(v, N)$ is the Shapley value of the ith position with v being a 2-additive measure on N.

Similar to additive distance measures listed in Subsection 3.1, we further define the generalized intuitionistic hesitant fuzzy linguistic 2-additive Choquet Shapley hybrid weighted distance (G2CSHWD$^{\mathrm{HIFL}}$) measure as follows:

Definition 11. Let $X=\{IH_1, IH_2, ..., IH_n\}$ and $Y=\{IH'_1, IH'_2, ..., IH'_n\}$ be any two sets of HIFLSs for the predefined linguistic term set $S=\{s_1, \ldots, s_t\}$. Then, the G2CSHWD$^{\mathrm{HIFL}}$ measure between X and Y is defined by

$$\mathrm{G2CSHWD}^{\mathrm{IHFL}}_{v,\mu}(X,Y)=\left(\sum_{i=1}^{n}\frac{Sh_i(v,N)Sh_{(i)}(\mu,D)}{\sum_{i=1}^{n}Sh_i(v,N)Sh_{(i)}(\mu,D)}d(IH_{(i)}, IH'_{(i)})^r\right)^{\frac{1}{r}},$$

where $r\in \mathrm{R}^+$, $(\cdot)$ is a permutation on $N=\{1, 2, \ldots, n\}$ such that $Sh_{(i)}(\mu,D)d(IH_{(i)}, IH'_{(i)})$ is the ith least value of $Sh_i(\mu,D)d(IH_i, IH'_i)$, $i\in N$, and $Sh_i(\mu, D)$ is the Shapley value of $d(IH_i, IH'_i)$ with μ being a 2-additive measure on $D=\{\, d(IH_i, IH'_i)\,\}_{i\in N}$, and $Sh_i(v,N)$ is the Shapley value of the ith position with v being a 2-additive measure on N.

Remark 1. When there are no interactions, the G2CSHWD$^{\mathrm{HIFL}}$ measure degenerates to the GHWD$^{\mathrm{HIFL}}$ measure. With respect to the different values of r, one can obtain the different intuitionistic hesitant fuzzy linguistic 2-additive Choquet Shapley hybrid weighted distance measures.

4. A method to intuitionistic hesitant fuzzy linguistic multi-attribute decision making

Consider a multi-attribute decision-making problem: suppose there are m alternatives $A=\{a_1, a_2, \ldots, a_m\}$ and n attributes $C=\{c_1, c_2, \ldots, c_n\}$. The evaluation of the alternative a_i with respect to the attribute c_j is an HIFLS IH_{ij} (i= 1, 2, …, m; j = 1, 2, …, n). By $H=\left(IH_{ij}\right)_{m\times n}$, we denote the HIFLS matrix given by a decision maker team.

If the weights are partly known, we first need to obtain their weight vectors. Let us give the concepts of score and accuracy functions as follows:

Definition 12. Let IH={<$s_{\theta(i)}$, ($\alpha_{(i)}$, $\beta_{(i)}$)> |$s_{\theta(i)}\in S$)} be an HIFLS for the predefined linguistic term set S={s_1, …, s_t}. The score function of IH is defined by $s(IH)=\sum_{s_{\theta(i)}\in S}\frac{s_{\theta(i)}(\alpha_{(i)}-\beta_{(i)})}{c(IH)t}$, and the accuracy function of IH is defined by

$h(IH)=\sum_{s_{\theta(i)}\in S}\frac{s_{\theta(i)}(\alpha_{(i)}+\beta_{(i)})}{c(IH)t}$, where $c(IH)$ is the number of linguistic terms in

IH.

From Definition 12, the ordered relationship, for any two HIFLSs IH_1 andIH_2, is defined by

If $s(IH_1) < s(IH_2)$, then $IH_1<IH_2$.

If $s(IH_1) = s(IH_2)$, then $\begin{cases} h(IH_1)=h(IH_2)\Rightarrow IH_1=IH_2 \\ h(IH_1)<h(IH_2)\Rightarrow IH_1>IH_2 \end{cases}$.

4.1. Models for the weight vectors

With respect to the HIFLS matrix $H=\left(IH_{ij}\right)_{m\times n}$, let $IH^+=\{IH_1^+,IH_2^+,...,IH_n^+\}$ be the positive ideal HIFLS set with $IH_j^+=\max_{1\le i\le m} IH_{ij}$, and let $IH^-=\{IH_1^-,IH_2^-,...,IH_n^-\}$ be the negative ideal HIFLS set with $IH_j^-=\min_{1\le i\le m} IH_{ij}$.

Since the weighted comprehensive distance measure $d(IH_{ij},IH_j^+)$ the smaller the better, and weighted comprehensive distance measure $d(IH_{ij},IH_j^-)$ the bigger the better, when the weight vector on the attribute set C is not exactly known, then we build the following model for the optimal 2-additive measure μ.

$$\min\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{d(IH_{ij},IH_j^+)}{d(IH_{ij},IH_j^+)+d(IH_{ij},IH_j^-)}Sh_{c_j}(\mu,C)$$

$$s.t.\begin{cases}\sum_{c_l\subseteq S\setminus c_j}\left(\mu(c_j,c_l)-\mu(c_l)\right)\geq(s-2)\mu(c_j),\ \ \forall S\subseteq C,\forall c_j\in S,\ s\geq 2\\ \sum_{\{c_j,c_l\}\subseteq C}\mu(c_j,c_l)-(n-2)\sum_{c_j\in C}\mu(c_j)=1\\ \mu(c_j)\in W_{c_j},\mu(c_j)\geq 0,\ \ j=1,2,...,n\end{cases}, \tag{9}$$

where W_{c_j}, j=1, 2, …, n, is the known weight information, s and n are cardinalities of coalitions S and N, respectively.

Furthermore, when the weights of the ordered positions are incompletely known, then we establish the following model for the optimal 2-additive measure v on the ordered set N={1, 2, …, n}.

$$\min\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{Sh_{c_{(j)}}(\mu,C)d(IH_{i(j)},IH_{(j)}^{+})}{d(IH_{i(j)},IH_{(j)}^{+})+d(IH_{i(j)},IH_{(j)}^{-})}Sh_j(v,N)$$

$$s.t.\begin{cases}\sum_{k\subseteq S\setminus j}\left(v(j,k)-v(k)\right)\geq(s-2)v(j),\ \ \forall S\subseteq N,\forall j\in S,\ j\geq 2\\ \sum_{\{j,k\}\subseteq N}v(j,k)-(n-2)\sum_{j\in N}v(j)=1\\ v(j)\in W_j,v(j)\geq 0,\ \ j=1,2,...,n\end{cases}, \tag{10}$$

where W_j, j=1, 2, …, n, is the known weight information, and $(\cdot)$ is a permutation on N such that $\frac{Sh_{c_{(j)}}(\mu,C)d(IH_{i(j)},IH_{(j)}^{+})}{d(IH_{i(j)},IH_{(j)}^{+})+d(IH_{i(j)},IH_{(j)}^{-})}$ is the jth least value of $\frac{Sh_{c_j}(\mu,C)d(IH_{ij},IH_j^{+})}{d(IH_{ij},IH_j^{+})+d(IH_{ij},IH_j^{-})}$ for j=1, 2, …, n.

4.2. A decision-making method

This subsection dedicates to give a method to multi-attribute decision making with intuitionistic hesitant fuzzy linguistic information. Based on the G2CSHWD$^{\mathrm{HIFL}}$ measure, the main procedure is described as follows:

Step 1: Assume that the evaluation of the alternative a_i with respect to the attribute

c_j is an HIFLS IH_{ij} from the predefined linguistic term set $S=\{s_q \mid q=0, 1, \ldots, t\}$. Let $H=\left(IH_{ij}\right)_{m\times n}$ be the HIFLS decision matrix. Transform $H=\left(IH_{ij}\right)_{m\times n}$ into $H'=\left(IH'_{ij}\right)_{m\times n}$, where $IH'_{ij}=\begin{cases} IH_{ij} & \text{for benefit attribute } c_j \\ \left(IH_{ij}\right)^c & \text{for cost attribute } c_j \end{cases}$ (i=1, 2, …, m; j=1, 2, …, n) with $\left(IH_{ij}\right)^c=\bigcup_{\langle s_{\theta(j)},(\alpha_{(j)},\beta_{(j)})\rangle\in IH_{ij}}\left\langle s_{t-\theta(j)},(\beta_{(j)},\alpha_{(j)})\right\rangle$.

Step 2: Use model (9) to solve the optimal 2-additive measure μ on the attribute set $C=\{c_1, c_2, \ldots, c_n\}$ and to calculate their Shapley values using Eq.(3).

Step 3: Utilize model (10) to solve the optimal 2-additive measure v on the ordered set $N=\{1, 2, \ldots, n\}$ and to calculate their Shapley values using Eq.(3).

Step 4: Use the G2CSHWD$^{\text{HIFL}}$ measure to calculate the comprehensive distance measures $D(IH'_i, IH^+)$ and $D(IH'_i, IH^-)$, where IH'_i is the ith row of H', and IH^+ and IH^- are respectively the positive ideal HIFLS set and the negative ideal HIFLS set for H'.

Step 5: Calculate the ranking indices R_i of alternatives a_i (i=1, 2, …, m), where

$$R_i=\frac{D(IH'_i,IH^+)}{D(IH'_i,IH^+)+D(IH'_i,IH^-)}\quad i=1, 2, \ldots, m.$$

According to R_i (i=1, 2, …, m), select the best choice.

Step 6: End.

Remark 2. When we calculate the distance measure between IH'_i and IH^-, it requires the permutation $(\cdot)$ on N = {1, 2, …, n} such that $Sh_{(i)}(\mu,D)d(IH'_{(i)},IH^-)$ is the ith largest value of $Sh_i(\mu,D)d(IH'_i,IH^-)$, $i\in N$.

5. An illustrative example

Let us consider the decision-making problem of assessing engines (adapted from Isitt, 1990). There are four brands of engine (alternatives) $A=\{a_1, a_2, a_3, a_4\}$ that are assessed using the linguistic term set $S=\{s_1$: poor, s_2: indifferent, s_3: average, s_4: good, s_5: excellent} with respect to four attributes: c_1: responsiveness, c_2: fuel economy, c_3: vibration, and c_4: starting. The assessment values given by the decision maker team are obtained as shown in Table 1.

Table 1. HIFLSs of the alternatives

	c_1	c_2
a_1	{<s_2, (0.2, 0.5)>, (<s_3, (0.4, 0.3)>}	{<s_4, (0.3,0.5)>, <s_5, (0.4, 0.2)>}
a_2	{<s_3, (0.3, 0.4)>, <s_4, (0.5, 0.2)>}	{<s_3, (0.6, 0.3)>, <s_4, (0.2, 0.5)>}
a_3	{<s_4, (0.6, 0.2)>, <s_5, (0.3, 0.5)>}	{<s_5, (0.4, 0.5)>}
a_4	{< s_3, (0.7, 0.2)>}	{<s_3, (0.3, 0.5)>, <s_4, (0.5, 0.3)>}
	c_3	c_4
a_1	{<s_3, (0.6, 0.3)>}	{<s_3, (0.2, 0.3)>, <s_4, (0.6, 0.2)>}
a_2	{<s_4, (0.7, 0.3)>}	{<s_3, (0.4, 0.5)>}
a_3	{<s_2, (0.3, 0.6)>, <s_3, (0.6, 0.2)>}	{<s_3, (0.7,0.3)>}
a_4	{<s_2, (0.6, 0.3)>, <s_3, (0.4, 0.3)>}	{<s_4, (0.6, 0.2)>, <s_5, (0.2, 0.4)>}

Assume that the importance of attributes is defined by

$$W_{c_1} = [0.2, 0.35], W_{c_2} = [0.25, 0.4], W_{c_3} = [0.2, 0.3], W_{c_4} = [0.15, 0.25]$$

and the importance of the ordered positions is given as

$$W_1 = [0.3, 0.4], W_2 = [0.25, 0.3], W_3 = [0.2, 0.25], W_4 = [0.1, 0.2].$$

Using the above procedure, the ranking indices are obtained as shown in Fig. 1.

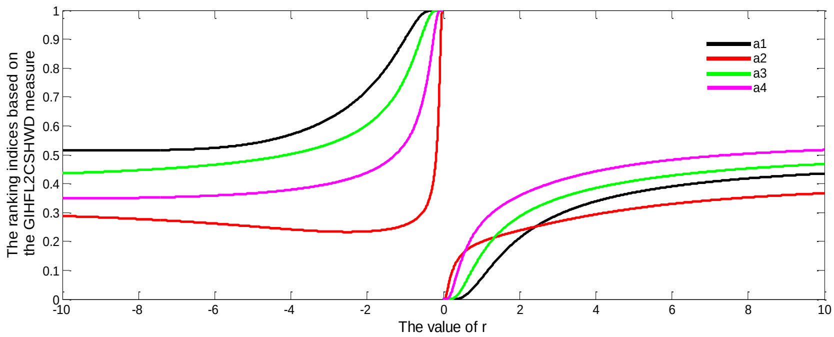

Figure 1. Ranking indices based on the G2CSHWD^{HIFL} measure for $r \in [-10, 0) \cup (0,10]$

Figure 1 shows that the alternative a_2 the best choice for $r \in [-10,0) \cup (2.436, 10]$. However, the alternative a_1 is the best choice for $r \in (0, 2.436]$.

In the above example, when we assume that there are no interactions between the attributes and between the ordered positions, by the GHWD^{HIFL} measure the ranking indices are derived as shown in Figure 2.

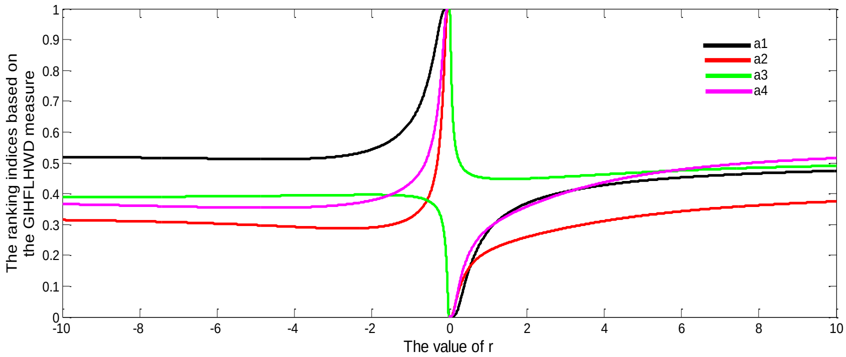

Figure 2. Ranking indices based on the GHWD^{HIFL} measure for $r \in [-10, 0) \cup (0,10]$

Figure 2 indicates that the alternative a_2 is the best choice for $r \in [-10,-0.573) \cup (0.542,10]$. However, the best choice is a_1 the alternative for $r \in (0,0.542]$, and the alternative a_3 is the best choice for $r \in (-0.573, 0)$.

By the G2CSHWD^{HIFL} and GHWD^{HIFL} measures, the different ranking orders are obtained. However, for $r \to -10$ or $r \to 10$, one can see that the alternative a_2 is the best choice. Furthermore, for $r \to 0^+$, the alternative a_1 is the best choice in these two cases.

From the illustrative example, we know that the different best choices may be obtained using the different distance measures and the different values of r, which requires the decision makers to determine the using distance measure and the value of r before making a decision. As one noted, the G2CSHWD^{HIFL} measure considers the interactions between elements in a set, and it can be seen as an extension of the GHWD^{HIFL} measure. Thus, when there are no special explanations that the considered elements are independent, we suggest the decision maker use the G2CSHWD^{HIFL} measure. As for r, it depends on the decision makers' risk attitudes.

5. Conclusion

To cope with decision-making problems with qualitative preferences, this paper defines a new kind of fuzzy sets called hesitant intuitionistic fuzzy linguistic sets (HIFLSs) that consider the hesitancy and uncertainty of decision makers as well as give the quantitative preference of each linguistic term. This kind of fuzzy sets gives the decision makers more choices to express their individual preferences. To research its application, a distance between HIFLSs is defined, by which some distance measures are defined. To address the situation where the weight information is not exactly known, models for the optimal 2-additive measures and additive measures are respectively built. Furthermore, a decision-making method is developed, and an illustrative example is given to show the concrete application of the proposed procedure.

However, we only research the application of HIFLSs in decision making, we can also use the introduced distance measures and models for the optimal weight vectors in other fields, such as industrial engineering, decision maker systems, neural networks, digital image processing, and uncertain systems and controls. Furthermore, it will be interesting to define some operational laws on HIFLSs, and then define some aggregation operators, which will further extend the application of HIFLSs.

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