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OPTIMAL CHANGE-LOSS REINSURANCE CONTRACT DESIGN UNDER TAIL RISK MEASURES FOR CATASTROPHE INSURANCE

Abstract. *In this paper, the optimal reinsurance contract design problem for catastrophe insurance is studied using the general structure of reinsurance contracts, change-loss reinsurance. Closed-form solutions are derived under two tail risk measures, Value-at-Risk (VaR) and Conditional Tail Expectation (CTE). The results show that CTE is a robust risk measure in that the structure of the optimal reinsurance contract under CTE measure is always change-loss reinsurance. While the optimal reinsurance contract under VaR measure degenerates from change-loss reinsurance to quota-share reinsurance when the ceding company is less risk averse. The theoretical approach is also applied to earthquake insurance market in China's Yunnan Province, and explicit solutions to the optimal reinsurance contract design problem under both VaR and CTE measures are obtained in the paper.*

Keywords: *Catastrophe Insurance; Change-Loss Reinsurance; Contract Design; Value-at-Risk; Conditional Tail Expectation.*

JEL Classification : G22, C61

1. Introduction

The problem of optimal reinsurance contract design has drawn significant interest since the fundamental work of Borch (1960) and Arrow (1963). Assuming that the reinsurance premium follows expected value principle, Borch (1963) showed that stop-loss reinsurance minimizes the variance of the retained loss, and Arrow (1963) proved that stop-loss reinsurance maximizes the expected utility of the terminal wealth of a risk-averse insurer. The following research can be mainly divided into three types, (a) extending the family of risk premiums, (b) extending the family of risk measures, and (c) extending both.

The first type of research is devoted to investigating the optimal reinsurance problem using different premium principles. Kaluszka (2001) considered mean-variance premium principle, and Kaluszka (2005) used convex premium principles (exponential, semi-deviation and semi-variance, etc.) in analysis, respectively. Tan *et al*. (2009) investigated 17 reinsurance premium principles and established the sufficient and necessary (or just sufficient) conditions for the existence of the nontrivial optimal reinsurance in several cases. Cheung (2010) and Chi and Tan (2013) extended the problem with a family of general risk premiums.

The second type of research is dedicated to studying the optimal reinsurance problem under different risk measures. Among them, the tail risk measures are the most popularly used in recent years. Cai and Tan (2007) calculated the optimal retention level for stop-loss reinsurance using the Value-at-Risk (VaR) and Conditional Tail Expectation (CTE) risk measures. Following Cai and Tan's work, A series of research has been focused on obtaining the optimal reinsurance contract under VaR measure or CTE measure using different reinsurance structures (See Bernard and Tian (2009), Chi and Tan (2011), Tan and Weng (2012), Tan and Weng (2014), Cai et al. (2014), etc.) Some other risk measures are also used in literature. For example, Chi and Lin (2014) studied the problem from the perspective of an insurer, where an upper limit is imposed on a reinsurer's expected loss over a prescribed level. Cheung *et al*. (2014) have extended the problem by using general law-invariant convex risk measures, and compared the optimal reinsurance contract with the ones under the VaR and CTE risk measures. Asimit *et al*. (2015) studied the optimal non-life reinsurance problem by minimizing the risk exposure under Solvency II regime. Liang and Yuen (2016) studied the optimal proportional reinsurance problems under the criterion of maximizing the expected exponential utility.

Some recent studies also investigated the problem by extending both the risk premiums and the risk measures. Cheung *et al*. (2012) considered the optimal reinsurance problem using Wang's premium principle subject to the insurer's budget constraint and the reinsurer's ruin probability constraint under CTE measure. Cui *et al.* (2013) discussed the optimal reinsurance problem with the insurer's risk measured by distortion risk measure and the reinsurance premium calculated by the Wang's premium principle. Cong and Tan (2014) analyzed VaR based optimal risk management solution using reinsurance under a class of premium principles that is monotonic and piecewise. Cheung and Lo (2015) followed Cui *et al.* (2013), and investigated the characteristics of the optimal reinsurance contract. Zhuang *et al*. (2016) combined the Marginal Indemnification Function (MIF) formulation and the Lagrangian dual method to solve optimal reinsurance model with distortion risk measure and distortion reinsurance premium principle.

Though existing research has considered many factors in the reinsurance contract design problem, none of them have dig into the specific heavy-tailed catastrophe insurance market. Unlike conventional insurance losses, catastrophe losses are always characterized by heavy tailed distributions, which makes the setup for catastrophe reinsurance different, not only in the premium principles, but also in risk measures. In this paper, we try to obtain the optimal reinsurance contract for catastrophe insurance using the general form of reinsurance contracts, change-loss reinsurance. (1) For premium principles, the heavy tail property of catastrophe losses makes the premium principles that include the calculation of variance or even higher moments of the losses, such as mean-variance premium principle, not applicable. Therefore, only the expected value principle can be used in the calculation of the premium. (2) For risk measures, the heavy tail property indicates that there are chances that some extremely large losses may occur, which may have huge impact on the insurance company's solvency level. Therefore, the insurance company should pay more attention to the tail risk. Two tail risk measures, Value-at-Risk (VaR) measure and Conditional Tail Expectation (CTE) measure are used in the paper in designing the optimal catastrophe reinsurance plans, and closed-form solutions are derived. The results show that CTE is a robust risk measure in that the structure of the optimal reinsurance contract under CTE measure is always change-loss reinsurance. While the optimal reinsurance contract under VaR measure degenerates from change-loss reinsurance to quota-share reinsurance when the ceding company is less risk averse. Application of the model in earthquake insurance market in China's Yunnan Province is also studied in this paper. Optimal reinsurance contracts are designed under both VaR measure and CTE measure, and explicit solutions are derived.

The paper is organized as follows. In section 2, the optimal reinsurance contract design problems using VaR and CTE measures are investigated into. In Section 3, an application of the model in earthquake insurance in China's Yunnan province is studied. Section 4 concludes the paper.

2. Optimal Catastrophe Reinsurance Contract Design

2.1. Reinsurance Contract

Let X denote the catastrophe event loss, and $g(X)$ denote the ceded loss function, and $P(g(X))$ be the corresponding reinsurance premium. The total risk exposure of the insurance company is $T(X) = X - g(X) + P(g(X))$. Under reinsurance premium constraint π, the goal of the insurance company is to minimize the total risk $\rho(T(X))$, where $\rho(\cdot)$ is a specific risk measure.

The optimization problem for the insurance company can be stated as below,

$$\min_{f} \rho\left(T(g(X))\right) \tag{1}$$

$$s.t. \quad P(g(X)) \leq \pi \tag{2}$$

$$0 \leq g(X) \leq X \tag{3}$$

Eq. (3) is a natural restriction of the optimization problem, the ceded loss cannot be larger than the loss incurred by the insurance company since the insurance company cannot make profit by ceding the losses.

To specify the optimization problem, three factors should be taken into consideration:

(1) The form of the reinsurance contract

Various types of reinsurance contracts have been developed. Two most commonly used types of reinsurance contracts are:

A) Stop-loss reinsurance

$$g(X) = (X - d)_+ \tag{4}$$

where $d \geq 0$ is the deductible.

B) Quota-share reinsurance

$$g(X) = cX \tag{5}$$

where $0 < c < 1$ is the proportitional parameter denoting the share the total losses the reinsurance company is covering.

In practice, reinsurance is always more complicated. A combination of stop-loss reinsurance and quota-share reinsurance, which is named as change-loss reinsurance, is frequently used. change-loss reinsurance is defined as below:

C) Change-loss reinsurance

$$g(X) = c(X - d)_+ \tag{6}$$

where $0 < c \leq 1$ is the proportitional parameter and $d \geq 0$ is the deductible.

Note that stop-loss reinsurance corresponds to the case $c = 1$ while quota-share reinsurance corresponds to the case $d = 0$. With condition $0 < c \leq 1; d \geq 0$, condition (3) can be satisfied.

The parameters c, d are commonly defined as constants in practice, while in some other cases, they can be also be variable. For example, the reinsurance company may want to avoid taking too much responsibility when a sufficiently large amount of loss occurs, thus the proportion of loss the reinsurance company is willing to cover, i.e. c, can be a decreasing function of loss X. In this paper, we only consider the case of constant parameters, other cases can be analyzed similarly.

(2) The calculation of the reinsurance premium

Assuming that the general premium principle applies here,

$$P((g(X)) = E(g(X)) + C(g(X)) \tag{7}$$

where $C(\cdot)$ is the cost function, including the fees and profits. The mean-variance premium principle assumes that the cost function is related to the standard deviation or variance of the ceded loss. Two commonly used forms of the mean-variance premium are,

$$C(g(X)) = \beta \cdot D(g(X)) \tag{8}$$

and

$$C(g(X)) = \beta \cdot D^2(g(X)) \tag{9}$$

where $D(\cdot)$ denotes the standard deviation function, and $D^2(\cdot)$ denotes the variance function.

However, neither of the two premium principles can be used in catastrophe insurance due to the non-existence of the variance caused by the heavy tail property. In catastrophe risk modeling, the heavy tail property of catastrophe losses makes the common distributions like the lognormal distribution, the Gamma distribution, or the Weibull distribution not applicable in fitting the loss data. Following Embrechts *et al.*(1997), the heavy-tailed distribution, Generalized Pareto Distribution (GPD), is used to fit the loss data.

Given threshold μ, the CDF of GPD is defined as,

$$F_X(x) = 1 - \left(1 + \xi \frac{x-\mu}{\alpha}\right)^{-\frac{1}{\xi}} \quad x > \mu, \xi \neq 0 \tag{10}$$

where ξ, α are the shape parameter and the scale parameter, respectively.

The variance of GPD exists if and only if $\xi < 1/2$, which can not be satisfied in catastrophe insurance.

Assuming that the cost function can be denoted as a proportion of the expectation of the ceded loss $C(g(X)) = \theta \cdot E(g(X))$, where θ is the loading factor. Therefore, the reinsurance premium can be derived as,

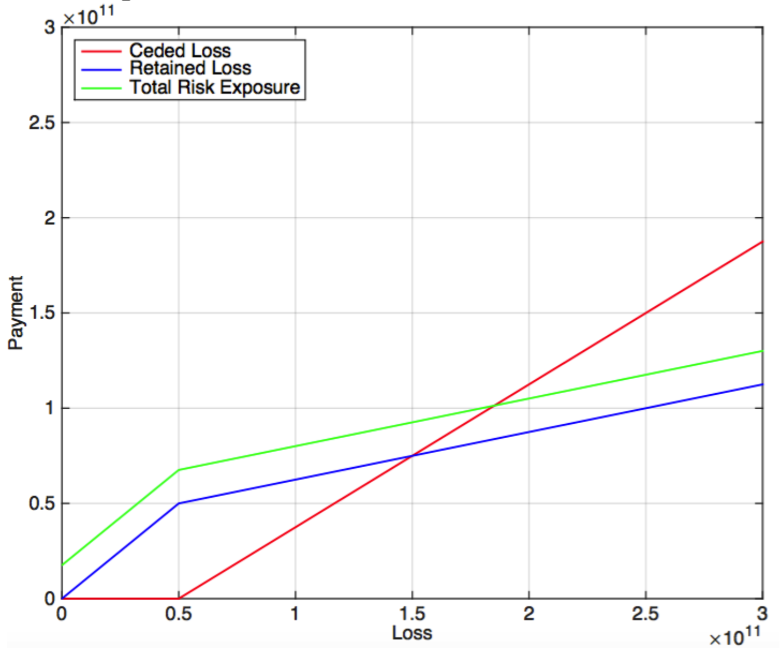
$$\begin{aligned} P(g(X)) &= (1+\theta) \cdot E(g(X)) \\ &= (1+\theta) \cdot c \cdot \left[\frac{\alpha + d\xi - \mu\xi}{1-\xi}\left(\frac{\alpha + d\xi - \mu\xi}{\alpha}\right)^{-\frac{1}{\xi}}\right] \end{aligned} \tag{11}$$

It can be see that the relationship between the reinsurance premium and the proportion parameter c is simply linear. The total risk exposure of the insurance company can be derived as,

$$\begin{aligned} T(g(X)) &= X - g(X) + P(g(X)) \\ &= X - c(X-d)_+ + (1+\theta) \cdot c \cdot \left[\frac{\alpha + d\xi - \mu\xi}{1-\xi}\left(\frac{\alpha + d\xi - \mu\xi}{\alpha}\right)^{-\frac{1}{\xi}}\right] \end{aligned} \tag{12}$$

With parameters $\theta = 0.2, c = 0.6, d = 5 \times 10^{10}$, and GPD parameters $\alpha = 5 \times 10^8, \xi = 0.98, \mu = 1 \times 10^8$, Relationship between the ceded loss, retained loss, and the total risk exposure for different loss amount X can be seen in Figure 1.

Figure 1: Ceded loss, retained loss and total loss



(3) The risk measure used by the insurance company

The heavy tail property of the catastrophe insurance losses reveals that there are chances that some extremely large losses may occur, which may even have huge effect on the solvency level. Therefore, the insurance company should pay more attention to the tail risk. To better measure the tail risk, the Value-at-Risk (VaR) measure and the Conditional Tail Expectation (CTE) measure should be used.

VaR of a random variable Z at confidence level $1-k$, $0<k<1$, is defined as a threshold loss value as below,

$$VaR_k(Z) = \inf z : Pr(Z \leq z) \geq 1-k \tag{13}$$

In practice, the value of k is typically very small, like $5\%, 2.5\%, 1\%, 0.1\%, etc$. By controlling the VaR of the loss payment, the insurance company can ensure with a high degree of confidence that the loss will not exceed the specific limit. Therefore, the VaR measure is able to reflect the tail risk of the loss, which is suitable for insurance companies that underwrites catastrophe insurance policies.

However, there are also some drawbacks of VaR. For example, VaR is not a coherent risk measure since it violates the sub-additivity property. The inequality $VaR_k(X) + VaR_k(Y) \leq VaR_k(X+Y)$ does not necessarily satisfy. In addition, VaR is also criticized about its inadequacy in capturing the tail behavior of the loss distribution since the risk measure does not pay attention to very extreme losses above a specific threshold.

The Conditional Tail Expectation (CTE), which is also called Tail Value-at-Risk, Average Value-at-Risk or Expected Shortfall, is a good alternative of

Value-at-Risk. CTE is defined as the conditional expectation of VaR with confidence level below some specific bound, denoted as below,

$$CTE_k(Z) = \frac{1}{k}\int_0^k VaR_\gamma(Z)d\gamma \tag{14}$$

Comparing to VaR, CTE is a coherent risk measure and it can better capture the tail risk. For a specific loss distribution, CTE is greater or equal to VaR at the same confidence level k, which helps the insurance company to pay attention to its extreme risks. However, the calculation of CTE is sometimes much more complex, and CTE is not that intuitive comparing to VaR. In practice, both VaR and CTE are widely used in estimating the tail risk, and both of the two measures are used to obtain the optimal reinsurance contract for the insurance company.

Substituting in Eq. (6), Eq. (11) and Eq. (12) into Eq. (1), Eq. (2) and Eq. (3), the optimization problem can be rewritten as,

$$\min_f \rho\left(X - c(X-d)_+ + (1+\theta)\cdot c\cdot\left[\frac{\alpha + d\xi - \mu\xi}{1-\xi}\left(\frac{\alpha + d\xi - \mu\xi}{\alpha}\right)^{-\frac{1}{\xi}}\right]\right) \tag{15}$$

$$s.t. \quad (1+\theta)\cdot c\cdot\left[\frac{\alpha + d\xi - \mu\xi}{1-\xi}\left(\frac{\alpha + d\xi - \mu\xi}{\alpha}\right)^{-\frac{1}{\xi}}\right] \le \pi \tag{16}$$

$$0 < \mathrm{c} \le 1; \quad d \ge 0 \tag{17}$$

where the risk measure $\rho(\cdot)$ can be either VaR or CTE.

2.2. Optimal reinsurance contract under VaR measure

The optimization problem can be specified by substituting Eq. (13) into Eq. (15) under the VaR measure. The VaR of total risk $T(X)$ at confidence level k can be derived as,

$$VaR_k(T(X)) = P(g(X)) + d + (1-c)\left[\frac{\alpha(k^{-\xi}-1)}{\xi} + \mu - d\right] \tag{18}$$

where $P(g(X))$ is calculated by Eq. (11).

Using method of Lagrange multipliers, the Lagrange function can be defined as,

$$\Lambda_1(c,d,\lambda) = VaR_k(T(X)) + \lambda\left\{(1+\theta)\cdot c\cdot\left[\frac{\alpha + d\xi - \mu\xi}{1-\xi}\left(\frac{\alpha + d\xi - \mu\xi}{\alpha}\right)^{-\frac{1}{\xi}}\right] - \pi\right\} \tag{19}$$

Take first order conditions of Eq. (19) over c, d, λ respectively, and let the derivatives equal to 0.

$$c - \frac{c\alpha^{1/\xi}(1+\theta)(1-\xi)(\alpha + d\xi - \mu\xi)^{-1/\xi}}{1-\xi}(1+\lambda) = 0 \tag{20}$$

$$d - \mu - \frac{(-1+k^{-\xi})\alpha}{\xi} + \frac{\alpha^{\frac{1}{\xi}}(1+\theta)(\alpha + d\xi - \mu\xi)^{1-\frac{1}{\xi}}}{1-\xi}(1+\lambda) = 0 \tag{21}$$

$$(1+\theta)\cdot c\cdot\left[\frac{\alpha+d\xi-\mu\xi}{1-\xi}\left(\frac{\alpha+d\xi-\mu\xi}{\alpha}\right)^{-\frac{1}{\xi}}\right]-\pi=0 \tag{22}$$

Solving the equations,

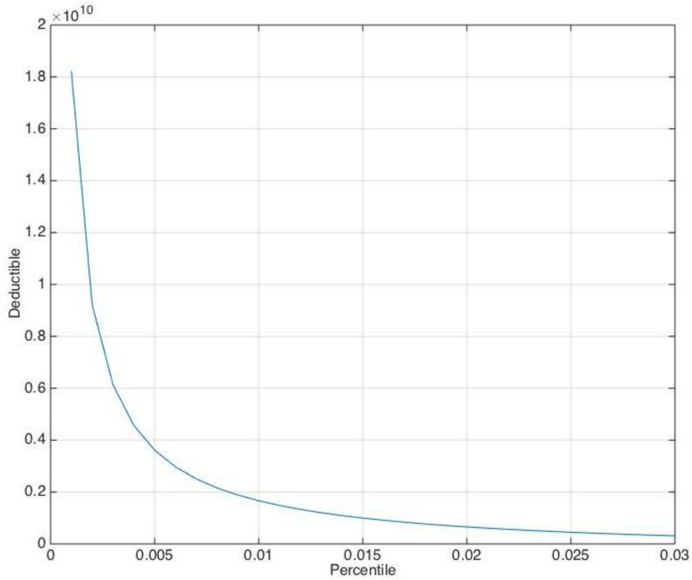
$$d^*=\frac{\alpha k^{-\xi}(1-\xi)+\mu\xi-\alpha}{\xi} \tag{23}$$

$$c^*=\frac{\pi(1-\xi)\alpha^{-\frac{1}{\xi}}}{1+\theta}(\alpha+d^*\xi-\mu\xi)^{-1+\frac{1}{\xi}} \tag{24}$$

From Eq. (23), it can be seen that the optimal deductible d^* of the reinsurance contract does not depend on the premium limit, it only depends on the parameters of the loss distribution, and the confidence level k of VaR measure.

Figure 2 shows the relationship between the optimal deductible and the confidence level. The loss distribution parameters are derived from the GPD model with $\alpha=5\times10^8, \xi=0.95, \mu=1\times10^8$. The premium limit is set as $\pi=2.5\times10^9$, and the loading is set as 0.2.

Figure 2: Relationship between d^* and k under VaR measure



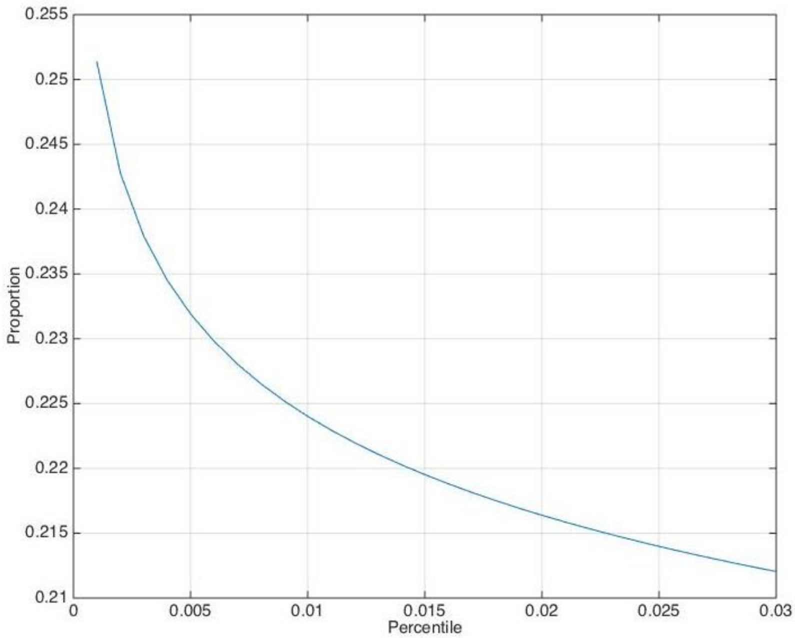
It can be seen from the figure that the deductible decreases when the confidence level k increases. For a more risk averse insurance company, say use $VaR_{0.01}$ instead of $VaR_{0.025}$, the amount of payments decreases to a large degree.

From Eq. (23), the optimal d^* is a constant for different VaR measures. The relationship between the proportion parameter c^* and the premium limit π is simply linear. To be more specific, c^* is linearly related to the pure premium limit, i.e. $\pi/(1+\theta)$. To see the relationship between c^* and confidence level k, substitute Eq. (23) into Eq. (24),

$$c^* = \frac{\pi(1-\xi)\alpha^{-\frac{1}{\xi}}}{1+\theta}\left[\alpha k^{-\frac{1}{\xi}}(1-\xi)\right]^{-1+\frac{1}{\xi}} = k^{-1+\xi} \cdot \frac{\pi(1-\xi)^{\frac{1}{\xi}}}{\alpha(1+\theta)} \tag{25}$$

Figure 3 shows the relationship between the optimal proportion c^* and the confidence level k.

Figure 3: Relationship between c^* and k under VaR measure



Comparing Figure 2 and Figure 3, it can be seen that when the confidence level k, changes, the change of the proportion c^* isn't that large as the deductible d^*. This can be proved by simply comparing $\partial c^*/\partial k$ and $\partial d^*/\partial k$. Intuitively, the heavy tail property of loss distribution has a larger impact on the loss percentiles, thus making the change of optimal deductible d^* larger. For a loss distribution with heavy tail property, the difference between the 90^{th} loss percentile ($VaR_{0.1}$) and the 99.9^{th} loss percentile ($VaR_{0.001}$) is much larger, thus the optimal deductible d^* for catastrophe reinsurance is much larger if the insurance company is more risk averse and uses $VaR_{0.001}$ instead of $VaR_{0.01}$ as the risk measure.

Note that the deductible d^* should be nonnegative, for a specific distribution $GPD(\mu, \sigma, \xi)$, with $\mu\xi - \alpha < 0$, the optimal deductible should be,

$$d^* = \max\left\{\frac{\alpha k^{-\frac{1}{\xi}}(1-\xi) + \mu\xi - \alpha}{\xi}, 0\right\} \tag{26}$$

The maximum confidence level that makes the optimal deductible d^* positive can be derived as,

$$\frac{\alpha k^{-\frac{1}{\xi}}(1-\xi)+\mu\xi-\alpha}{\xi}>0 \Rightarrow k<\left[\frac{\alpha-\mu\xi}{\alpha(1-\xi)}\right]^{-1/\xi}=\bar{k} \tag{27}$$

For $k>\bar{k}$, the optimal parameters c^*, d^* can be derived as,

$$d^*=0 \tag{28}$$

$$c^*=\frac{\pi(1-\xi)\alpha^{-\frac{1}{\xi}}}{1+\theta}[\alpha-\mu\xi]^{-1+\frac{1}{\xi}} \tag{29}$$

With the parameters defined above, $\bar{k}=0.0533$, and correspondingly $d^*=0$ and $c^*=0.206$.

Note that, with the confidence level greater than a specific level $\bar{k}$, the optimal reinsurance contract remains the same, and is only related to the loss distribution (GPD) parameters, and the pure premium $\pi/(1+\theta)$.

To summarize, the optimal contract under VaR measure is a change-loss reinsurance for small confidence level k, and a quota-share reinsurance for large confidence level k.

2.3. Optimal reinsurance contract under CTE measure

Under the CTE measure, the optimization problem can be specified by substituting Eq. [14] into Eq. [15]. The CTE of total risk $T(X)$ at confidence level k can be derived as,

$$CTE_k(T(X))=P(g(X))+cd+(1-c)\mu-\alpha\frac{1-c}{\xi}\left(1-\frac{k^{-\xi}}{1-\xi}\right) \tag{30}$$

where $P(g(X))$ is calculated by Eq. (11).

Using method of Lagrange multipliers, the Lagrange function can be defined as,

$$\Lambda_2(c,d,\lambda)=CTE_k(T(X)) \\ +\lambda\left\{(1+\theta)\cdot c\cdot\left[\frac{\alpha+d\xi-\mu\xi}{1-\xi}\left(\frac{\alpha+d\xi-\mu\xi}{\alpha}\right)^{-\frac{1}{\xi}}\right]-\pi\right\} \tag{31}$$

Take F. O. C. over c, d, λ respectively, and let the derivatives equal to 0.

$$c-\frac{c\alpha^{\frac{1}{\xi}}(1+\theta)(1-\xi)(\alpha+d\xi-\mu\xi)^{-\frac{1}{\xi}}}{1-\xi}(1+\lambda)=0 \tag{32}$$

$$d-\mu+\frac{\alpha}{\xi}\left(1-\frac{k^{-\xi}}{1-\xi}\right)+\frac{\alpha^{\frac{1}{\xi}}(1+\theta)(\alpha+d\xi-\mu\xi)^{1-\frac{1}{\xi}}}{1-\xi}(1+\lambda)=0 \tag{33}$$

$$(1+\theta)\cdot c\cdot\left[\frac{\alpha+d\xi-\mu\xi}{1-\xi}\left(\frac{\alpha+d\xi-\mu\xi}{\alpha}\right)^{-\frac{1}{\xi}}\right]-\pi=0 \tag{34}$$

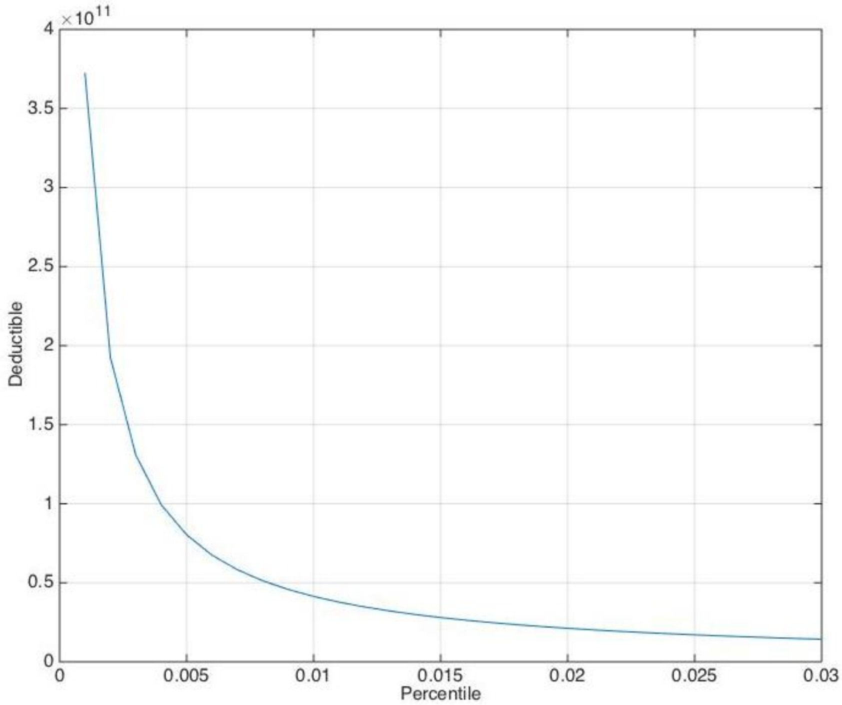
Solving the equations,

$$d^*=\frac{\alpha k^{-\xi}+\mu\xi-\alpha}{\xi} \tag{35}$$

$$c^* = \frac{\pi(1-\xi)\alpha^{-\frac{1}{\xi}}}{1+\theta}(\alpha + d^*\xi - \mu\xi)^{-1+\frac{1}{\xi}} \tag{36}$$

From Eq. (34), it can be seen that the optimal deductible d^* of the reinsurance contract does not depend on the premium limit, it only depend on the parameters of the loss distribution, and the confidence level k of CTE measure. Figure 4 shows the relationship between the optimal deductible d^* and the confidence level k, with the GPD parameters defined same as before.

Figure 4: Relationship between d^* and k under CTE measure



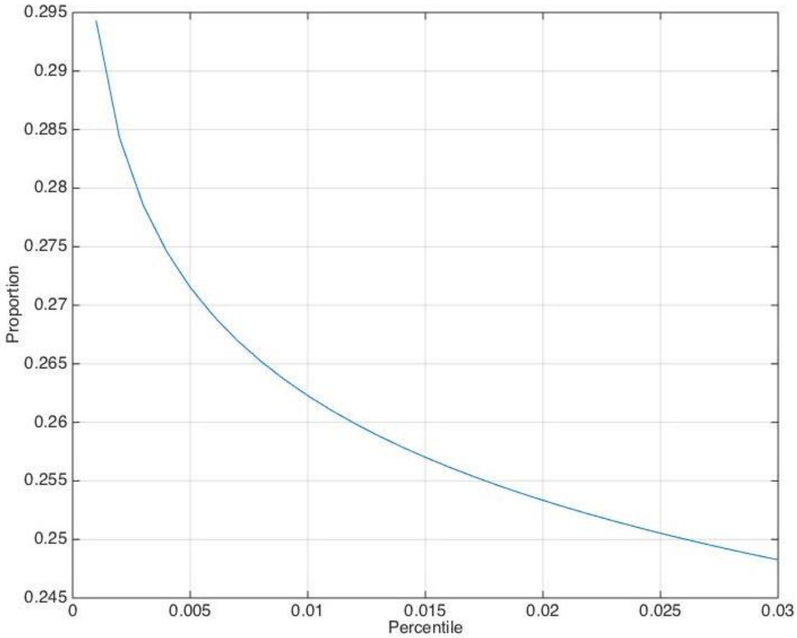
It can be seen from Figure 4 that the deductible decreases when the confidence level k increases. When the insurance is more risk averse, say use $CTE_{0.01}$ instead of $CTE_{0.025}$, the amount of payments decreases to a large extent.

Similar to the VaR case, the proportion parameter c^* is linearly related to the pure premium limit, i.e. $\pi/(1+\theta)$. To see the relationship between c^* and confidence level k, substitute Eq. (35) into Eq. (36),

$$c^* = \frac{\pi(1-\xi)\alpha^{-\frac{1}{\xi}}}{1+\theta}\left(\alpha k^{-\xi}\right)^{-1+\frac{1}{\xi}} = k^{-1+\xi} \cdot \frac{\pi(1-\xi)}{\alpha(1+\theta)} \tag{37}$$

Figure 5 shows the relationship between the optimal proportion c^* and the confidence level k.

Figure 5: Relationship between c^* and k under CTE measure



Comparing Figure 4 and Figure 5, it can be seen that when the confidence level changes, the change rate of the proportion parameter c^* isn't that large as the deductible parameter d^*, which is similar to the case of VaR measure.

Note that the confidence level $k \in (0,1)$, thus $k^{-\xi} > 1$, and

$$d^* = \frac{\alpha k^{-\xi} + \mu\xi - \alpha}{\xi} > \frac{\alpha + \mu\xi - \alpha}{\xi} = \mu > 0 \tag{38}$$

Therefore, the optimal deductible d^* is always positive, indicating that the optimal reinsurance under CTE measure is always a change-loss reinsurance, indicating that CTE is a robust risk measure in this case.

2.4. Comparison between VaR measure and CTE measure

Comparing the optimal contract under VaR measure and under CTE measure, it can be seen that with other parameters fixed, larger deductible parameter d^* means that the insurance company is more risk averse, while larger proportion parameter c^* means that the insurance company is less risk averse. To see the difference between the optimal reinsurance contracts under two tail risk measure, the optimal parameters c^*, d^* under VaR measure and CTE measure are listed in Table 1 with the parameters set as $\theta = 0.2$, $\pi = 2.5 \times 10^9$, and GPD parameters $\alpha = 5 \times 10^8, \mu = 1 \times 10^8, \xi = 0.98$.

Table 1: Comparison between VaR Measure and CTE Measure

k	0.001	0.005	0.010	0.025	0.050	$\overline{k}$
d_1	3.74×10^{10}	6.53×10^{9}	2.93×10^{9}	8.52×10^{8}	1.90×10^{8}	0
c_1	0.261	0.239	0.230	0.218	0.210	0.206
d_2	3.72×10^{11}	8.93×10^{10}	4.14×10^{10}	1.71×10^{10}	8.64×10^{9}	8.10×10^{9}
c_2	0.294	0.272	0.263	0.251	0.242	0.241

where c_1, d_1 stands for the optimal parameters c^*, d^* under VaR measure, and c_2, d_2 stands for the optimal parameters c^*, d^* under CTE measure.

From Table 1, it can be seen that the deductible d under CTE measure is much larger than that in VaR measure, while the proportion parameter c is also larger under CTE measure. Therefore, the insurance company is much more risk averse under CTE measure, requiring the reinsurance company takes more responsibility under CTE measure. In addition, the optimal reinsurance contract under VaR measure changes from a quota share to stop-loss reinsurance after a specific confidence level, and remains unchanged ever since. While under CTE measure, the optimal reinsurance contract is always a change-loss reinsurance contract, indicating that CTE is a robust risk measure for optimal reinsurance design.

3. Application

3.1. Data

China has been suffering catastrophes frequently in recent years, making the research on China's catastrophe insurance and reinsurance market of great importance. In this section, we use the earthquake loss data in China's Yunnan province as an example to find the optimal reinsurance contract. The loss data covers all the losses of earthquakes with magnitude above 4.0 occurred between 1978 and 2008. The data are collected from China's Earthquake Yearbook (1978-2008).

The economic loss of the earthquakes is strongly related to the economic status of the year, making it inappropriate to use the data directly to fit the loss data with a specific loss distribution. To eliminate the effect of the economic development, we define the modification factor as below.

$$factor_i = \frac{GDP_{2016}}{GDP_i} \tag{39}$$

where $i = 1978, 1979, \ldots 2008$, GDP_{2014} is the GDP of year 2014, and GDP_i denotes the GDP of year i. Here we use GDP_{2014} because GDP of more recent years, like 2015, haven't been released yet.

The modification factor of year 2014 is set as 1, and $factor_i$ is the modification factor of year i. By defining the modification factor, the losses in different years are all adjusted to the economic level of year 2014.

The factors for years 1978-2008 can be seen in Table 2.

Table 2: Modification Factors

Year	Factor	Year	Factor	Year	Factor
1978	185.45	1979	166.79	1980	152.01
1981	136.14	1982	116.37	1983	106.73
1984	91.81	1985	77.66	1986	70.30
1987	55.95	1988	42.56	1989	35.30
1990	28.35	1991	24.77	1992	20.71
1993	16.36	1994	13.03	1995	10.48
1996	8.44	1997	7.65	1998	7.00
1999	6.75	2000	6.37	2001	5.99
2002	5.54	2003	5.01	2004	4.16
2005	3.69	2006	3.20	2007	2.69
2008	2.25				

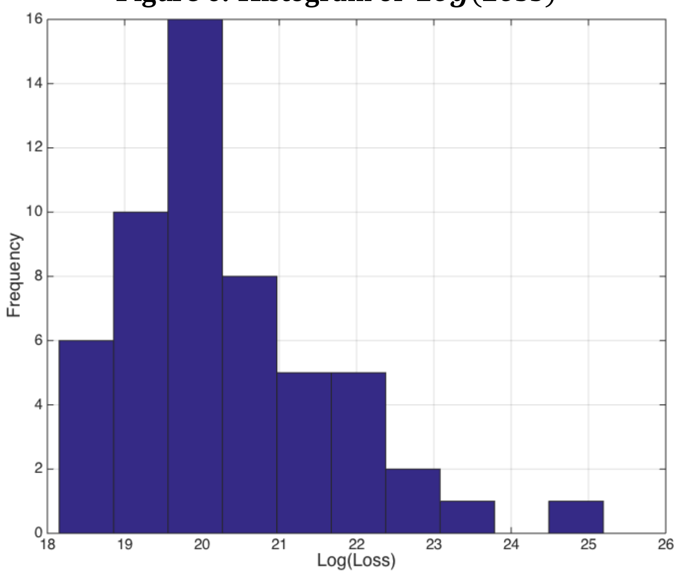
The modified economic loss can be derived by the direct economic loss multiplied by the modification factors, which is presented below:

$$Loss_i = OriginalLoss_i \times factor_i \tag{40}$$

where $OriginalLoss_i$ and $Loss_i$ are the source data from the yearbooks and the modified data.

The histogram of the logarithm of the modified loss data $Log(Loss)$ is shown in Figure 6.

Figure 6: Histogram of $Log(Loss)$



It can be seen from Fig. 6 that the losses are of heavy tail. The summary statistics of the losses are shown in Table 3.

Table 3: Summary Statistics of Losses

	Mean	Std. Dev.	Mean	Minimum	Maximum
Loss	3.06×10^7	1.21×10^{10}	4.83×10^8	7.65×10^7	8.72×10^{10}

The Coefficient of Variation (CV) of the losses can be derived as 3.95, indicating that the losses are of heavy tail. Using the Generalized Pareto Distribution (GPD) to fit the data, the parameters can be derived using Mean Excess Function (MEF) and Maximum Likelihood Estimation (MLE),

$$\mu = 8 \times 10^7,\ \alpha \approx 4.58 \times 10^8,\ \xi \approx 0.98 \tag{41}$$

3.2. Optimal reinsurance contract design

The variance of GPD exists iff $\xi < \frac{1}{2}$, for earthquake insurance in Yunnan province, the variance doesn't exist, making risk measures involving the variance not applicable. In this section, we try to find the optimal reinsurance contract under VaR and CTE.

(1) VaR measure

As is shown in Section 2.3, the optimal reinsurance contract has different forms for different confidence levels. The maximum confidence level $\bar{k}$ that makes the optimal deductible d^* positive can be derived by substituting the Eq. (41) into Eq. (27),

$$\bar{k} \approx 0.045 \tag{42}$$

For confidence levels $k < \bar{k}$, the optimal deductible d^* and optimal proportion c^* can be derived by substituting Eq. (41) into Eq. (23) and Eq. (24), while for confidence levels $k > \bar{k}$, the optimal deductible d^* is always 0, and the the optimal proportion c^* is also fixed, and can be derived by substituting Eq. (41) into (29). Table 4 shows the optimal contract for some confidence levels under VaR measure.

Table 4: Optimal Reinsurance Contract under VaR Measure

k	0.001	0.005	0.010	0.025	$\bar{k}$	0.050
d	5.78×10^9	1.11×10^9	4.95×10^8	1.16×10^7	0	0
c	0.482	0.453	0.441	0.426	0.417	0.417

(2) CTE measure

The optimal reinsurance contract under CTE measure can be derived by substituting Eq. (41) into Eq. (35) and Eq. (36). Table 5 shows the optimal contract for some confidence levels under VaR measure.

Table 5: Optimal Reinsurance Contract under CTE Measure

k	0.001	0.005	0.010	0.025	$\bar{k}$	0.050
d	4.04×10^{11}	8.33×10^{10}	4.21×10^{10}	1.69×10^{10}	9.40×10^{9}	8.40×10^{9}
c	0.484	0.468	0.461	0.452	0.447	0.446

Comparing Table 4 and Table 5, it can be seen that for a specific confidence level k, the optimal contract under CTE measure not only has a larger dedutible, but also cedes a larger proportion of the losses, comparing to the optimal contract under VaR measure. Therefore, the insurance company is much more risk averse under the CTE measure.

4. Conclusions

Optimal reinsurance design is an important topic in insurance research, especially in catastrophe reinsurance. Comparing to conventional insurance, the catastrophe insurance are of severe tail risk due to the heavy tail characteristic of the catastrophe losses, making optimal reinsurance design problem of greate significance. In this paper, the optimal reinsurance contract for catastrophe insurance is designed under two tail risk measures, VaR and CTE, with the catastrophe insurance losses characterized by the Generalized Pareto Distribution (GPD) and the reinsurance premium calculated using the expected value premium principle. The results show that under VaR measure, the optimal reinsurance contract is change-loss reinsurance for small confidence level, and quota-share reinsurance for large confidence level. Under CTE measure, the optimal reinsurance contract is always change-loss reinsurance. Comparing to the optimal reinsurance contract under VaR measure, the optimal insurance contract under CTE measure has a much larger deductible and also a larger ceded proportion, indicating that the insurance company is much more risk averse under the CTE measure for the same confidence level. We also applied the approach to earthquake insurance market in China's Yunnan Province, optimal reinsurance under both VaR measure and CTE measure are designed using the general reinsurance structure, change-loss reinsurance.

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