8 ON INSURER PORTFOLIO OPTIMIZATION. AN UNDERWRITING RISK MODEL

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Abstract

Multicriteria portfolio optimization started with the Markowitz mean-variance model (Markowitz 1952, 1959). This model assumes that the goal of an average or standard investor is to maximize the unknown return on investment. In this paper we propose a risk model related to insurance industry. The optimality criteria we propose for insurer's portfolio optimization are based on the well-known Markowitz model, yet imposing scalarization on the components of the objective function.

Key words: portfolio optimization, underwriting risk, scalarization.

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1. Introduction

Multicriteria portfolio optimization started with the Markowitz mean-variance model (Markowitz 1952, 1959). This model assumes that an average or standard investor seeks to maximize the unknown return on investment. The Markowitz model, which is considered the classical approach to portfolio optimization, is based on two opposed optimization criteria: firstly, the risk of a portfolio – which could be measured, for example, by its variance, should be minimized, and, then, the expected return of the portfolio has to be maximized. A possible deterministic equivalent to the mean-variance model is the stochastic optimization problem, with the objective to maximize the expected return subject to a constraint on its variance. In this context, an efficient solution is a portfolio which has the property that when moving to a portfolio with higher return, variance will also increase, and when moving to a portfolio with smaller variance, return will decrease too.

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In what follows we apply some scalarization techniques on a risk model related to insurance industry. The optimality criteria we propose for insurer's portfolio optimization are based on the well-known Markowitz model, yet imposing, as previously mentioned, classical and new techniques of scalarization on the objective function. In Section 2, we remind some definitions and the scalarization methods, while in Section 3, we present an insurer portfolio optimization model based on a study of Schnieper (2000). In Section 4 we apply the scalarization methods presented in Section 2 for the insurance company simple model, similar to the classical Markowitz model. Moreover, following the lines from Engau and Wicek (2005) and other authors (Preda and Sudradjat 2006, 2007), we prove some results concerning the relationships between optimal portfolios generated by the scalarized models and optimal portfolios generated by the initial model. The paper concludes to some final remarks presented in Section 5.

2. Definitions and scalarization methods

Let $D \subset \mathbf{R}^m$ be a *feasible set of portfolios* and f be a vector-valued objective function $f : \mathbf{R}^m \to \mathbf{R}^q$ composed of q real-valued objective functions, $f = (f_1, .., f_q)$, where $f_i : \mathbf{R}^m \to \mathbf{R}$, $i = 1, ..., q$. For $y, y' \in \mathbf{R}^m$. $y > y'$ denotes $y_i > y_i'$ for all $i = 1, .., m$. $y \geqq y'$ denotes $y_i \geq y_i'$ for all $i = 1, .., m$. $y \geq y'$ denotes $y \geqq y'$ but $y \neq y'$. The relations $\leqq$, $\leq$ and < are defined in the obvious way. Let $\mathbf{R}^m_{\geqq} = \{ y \in \mathbf{R}^m \mid y \geqq 0 \}$. The sets $\mathbf{R}^m_{\geq}$, $\mathbf{R}^m_{>}$ are defined accordingly.

The vector portfolio problem is given by

$$(VPP) : \min_{\alpha \in D} \left(f_1(\alpha), ..., f_q(\alpha) \right),$$

where the minimization is understood as finding the set of efficient solutions in D.

Definition 2.1. Consider the (VPP). A feasible portfolio $\hat{\alpha} \in D$ is called

(i) a *weakly efficient portfolio* if there does not exist $\alpha \in D$ such that $f(\alpha) < f(\hat{\alpha})$;

(ii) an *efficient portfolio* if there does not exist $\alpha \in D$ such that $f(\alpha) \leq f(\hat{\alpha})$.

A feasible portfolio $\alpha \in D$ is evaluated by the q objective functions producing the outcome $f(\alpha)$.

The set of all attainable outcomes for all feasible portfolios in the objective space is denoted by

$$Y = f(D) \subset \mathbf{R}^q .$$

The image $f(\alpha) \in Y$ of a (weakly) efficient portfolio is called *a (weak) Pareto outcome*.

Definition 2.2. Consider the (VPP) and let $\varepsilon \in \mathbf{R}^q_{\geqq}$. A point $\hat{\alpha} \in D$ is called a *properly efficient portfolio* if

(i) $\hat{\alpha}$ is an efficient portfolio for (VPP)

and

(ii) there exists $M > 0$ such that for each $i \in \{1,..,q\}$ and $\alpha \in D$ with $f_i(\alpha) < f_i(\hat{\alpha})$ and $j \in \{1,..,q\}$ that verifies $f_j(\hat{\alpha}) < f_j(\alpha)$, we have $\frac{f_i(\hat{\alpha}) - f_i(\alpha)}{f_j(\alpha) - f_j(\hat{\alpha})} \leqq M$.

Given the (VPP), one can formulate a *scalarized* (single objective) *portfolio problem* (SPP).

Let $S \subseteq D$ be a subset of the feasible set D, U a set of auxiliary variables and Π a set of parameters, $T = f(S)$ denote the set of attainable outcomes for the (SPP) and

$$s : T \times U \times \Pi \to \mathbf{R}$$

be a *scalarizing function*. Then the (SPP) associated with the (VPP) is given by

$$(SPP) : \min_{\substack{\alpha \in S \\ u \in U}} s(f(x), u, \pi),$$

where $\pi \in \Pi$ is a vector of parameters chosen by the decision maker.

Definition 2.3. Consider the (SPP). A point $(\hat{\alpha}, \hat{u}) \in S \times U$ is called

(i) *an optimal portfolio* if $s(f(\hat{\alpha}), \hat{u}, \pi) \leqq s(f(\alpha), u, \pi)$ for all $(\alpha, u) \in S \times U$; the outcome $\hat{y} = f(\hat{\alpha}) \in T$ is called optimal;

(ii) *strictly optimal portfolio* if $s(f(\hat{\alpha}), \hat{u}, \pi) < s(f(\alpha), u, \pi)$ for all $(\alpha, u) \in S \times U$; the outcome $\hat{y} = f(\hat{\alpha}) \in T$ is called strictly optimal.

The *scalarization methods* which we took into account in Section 4 are the following:

a) *Weighted-sum scalarization.* Here $S = D$, $T = Y$, $U = \Phi$ and $\Pi = \mathbf{R}^q_{\geqq}$;

$s : Y \times \mathbf{R}^q_{\geqq} \to \mathbf{R}$, $s(f(\alpha), w) = \sum_{i=1}^{q} w_i f_i(\alpha)$, where $w = (w_1, ..., w_q)$.

b) *Constrained-objective scalarization.* Here $S = D$, $T = \{y \in Y | y_i \leqq \delta_i, i = 2,..,q\}$, $U = \Phi$ and $\Pi = \mathbf{R}^{q-1}$; $\delta_i \in \mathbf{R}$, $i = 2,..,q$; $s : T \times \mathbf{R}^{q-1} \to \mathbf{R}$, $s(f(\alpha), \delta) = f_1(\alpha)$, $\delta = (\delta_2, .., \delta_q)$.

c) *Guddat scalarization.* Here $S = D$, $T = \left\{ y \in Y \middle| y_i \leqq f_i\left(\alpha^0\right), i = 1,..,q \right\}$, $U = \Phi$ and $\Pi = D \times \mathbf{R}^q_{\geqq}$, $\alpha^0 \in D$; $s : T \times \left(D \times \mathbf{R}^q_{\geqq} \right) \to \mathbf{R}$, $s\left(f(\alpha), \left(\alpha^0, w\right) \right) = \sum_{i=1}^{q} w_i f_i(\alpha)$.

d) *Benson scalarization* (this is a particular case of Guddat scalarization). Here $S = D$, $T = \left\{ y \in Y \middle| y \leqq y^0, i = 1,..,q \right\}$, $U = \Phi$, $\Pi = Y$, $y^0 = \left(f_1\left(\alpha^0\right), ..., f_q\left(\alpha^0\right) \right)$, $\alpha^0 \in D$; $s : T \times D \to \mathbf{R}$, $s\left(f(\alpha), \alpha^0 \right) = \sum_{i=1}^{q} f_i(\alpha)$.

e) *Min-max scalarization* (this is a special case of Tchebycheff-norm scalarization). Here $S = D$, $T = Y$, $U = \Pi = \Phi$; $s : Y \to \mathbf{R}$, $s\left(f(\alpha) \right) = \max_{1 \le i \le q} f_i(\alpha)$.

f) *Tchebycheff-norm scalarization.* Here $S = D$, $T = Y$, $U = \Phi$, $\Pi = \mathbf{R}^q \times \mathbf{R}^q_{\geqq}$; $s : Y \times \left(\mathbf{R}^q \times \mathbf{R}^q_{\geqq} \right) \to \mathbf{R}$, $s\left(f(\alpha), (r, w) \right) = \max_{1 \le i \le q} w_i \left(f_i(\alpha) - r_i \right)$.

g) *Gasimov scalarization.* $S = D$, $T = Y$, $U = \Phi$, $\Pi = \mathbf{R} \times \mathbf{R}^q$, $s : Y \times \left(\mathbf{R} \times \mathbf{R}^q \times \mathbf{R}^q_{\geqq} \right) \to \mathbf{R}$, $s\left(f(\alpha), (a, r, w) \right) = q \sum_{i=1}^{q} \left| f_i(\alpha) - r_i \right| + \sum_{i=1}^{q} w_i \left(f_i(\alpha) - r_i \right)$.

3. Schnieper's underwriting risk model

The profit and loss account of an insurance company typically details items like earned premiums (net of reinsurance), investment income and realized capital gains, and expenditure positions as incurred claims (net of reinsurance recoveries), expenses, dividends to policyholders, dividends to shareholders. Moreover, the premium is divided into its different components: *pure risk premium, loading for expenses, loading for profit.*

As in Schneiper (2000), in what follows we will assume that: expenses and loading for expenses are identical and therefore cancelled out; dividends to policyholders are accounted for as claims; also we could ignore the dividends to shareholders; the period under consideration is the financial year of the company; payments pertaining to a given period are made at the end of the period; the premium written in a given period is earned in that period, i.e. the company has no unearned premium reserves.

Moreover we consider the following *model assumptions* (Schneiper (2000)): all random variables appearing in the model have finite second order moments; the pure risk premium is the present value of the expected loss payments; the loss reserves are equal to the present values of expected future loss payments; the discount factors used to assess the pure risk premium and the loss reserves are based on the yield curve as defined by the bond market; the assets of the company are assessed at market value.

The notation used here are the same as in Schnieper (2000): $\tilde{S}$ stands for the *total claims amount* pertaining to the current accident year; $E(\tilde{S})$: the mathematical expectation of the total claims amount, that is the *pure risk premium*; ℓ is the *profit loading* for assuming the underwriting risk $\tilde{S}$; $\tilde{\Delta}L$ means *increase in claim amounts* in respect of claims pertaining to previous accident years; $\tilde{\Delta}A$ is *investment income plus realized capital gains plus unrealized capital gains*; u represents the *capital* (economic value) of the company at the beginning of the financial year; $\tilde{\Delta}u$ is the *increase in capital* (in economic value) during the financial year (return of the company during the financial year).

Obviously, we have the following relation:

$$\tilde{\Delta}u = E(\tilde{S}) + \ell - \tilde{S} - \tilde{\Delta}L + \tilde{\Delta}A,$$

and $\tilde{S} - E(\tilde{S})$ stands for *underwriting risk*, $\tilde{\Delta}L - E(\tilde{\Delta}L)$ for *loss reserve risk*, $\tilde{\Delta}A - E(\tilde{\Delta}A)$ is *the asset risk* and $\tilde{\Delta}u - E(\tilde{\Delta}u)$ is *the total risk of the company*.

In what it follows we discuss an underwriting risk, considering that the assets of the company are split between liability fund and capital fund,

$$A = A_L + A_U .$$

This means that some of the assets, A_L , cover the liabilities of the company, and the rest of the assets, A_U , match the equity of the company. Moreover, we assume that there is no loss reserve risk (amount and time of payment in respect to outstanding losses are perfectly known to the company) and no asset risk. Therefore, the liability fund, i.e. those assets which cover the liabilities perfectly match the amounts and maturities of the liabilities. The liabilities are discounted with the discount factors corresponding to the liability fund. As a consequence, any change in the yield curve will have a perfectly offsetting effect on $\tilde{\Delta}L$ and $-\tilde{\Delta}A_L$ and the capital fund is invested in the risk free rate of return: $\tilde{\Delta}A_U = \rho_0 u$ (Schnieper, 2000).

Now, the total return of the company is given by

$$\tilde{\Delta}u = E(\tilde{S}) + \ell - \tilde{S} - \tilde{\Delta}L + \tilde{\Delta}A_L + \tilde{\Delta}A_U = E(\tilde{S}) + \ell - \tilde{S} + \rho_0 u .$$

The objective is to provide a method to optimize the portfolio of the company. Thus, the company considers the *excess return on equity* provided by the insurance portfolio

$$\tilde{\delta}(u) = \frac{\tilde{\Delta}u - \rho_0 u}{u} = \frac{E(\tilde{S}) + \ell - \tilde{S}}{u} .$$

We consider $\tilde{S} = \sum_{i=1}^{m} \tilde{X}_i$, where $\tilde{X}_i$, $i \in \overline{1,m}$, are m individual risks (policies, lines of business).

The company manages its portfolio by defining for each risk $\tilde{X}_i - E(\tilde{X}_i)$ the share $\alpha_i \in [0,1]$ it wants to retain and by ceding $(1-\alpha_i)(\tilde{X}_i - E(\tilde{X}_i))$ to its reinsurers. It is assumed that the company also cedes a proportional share of the corresponding profit $(1-\alpha_i)\ell_i$ to its reinsurers (Schnieper, 2000).

The return of the net retained portfolio is thus

$$\tilde{\Delta}u_{\text{net}} = \sum_{i=1}^{m} \alpha_i \left(E(\tilde{X}_i) + \ell_i - \tilde{X}_i\right) + \rho_0 u$$

and the corresponding *excess return on equity* is

$$\tilde{\delta}_u(\alpha) = \frac{\tilde{\Delta}u_{\text{net}} - \rho_0 u}{u} = \sum_{i=1}^{m} \alpha_i \frac{E(\tilde{X}_i) + \ell_i - \tilde{X}_i}{u}.$$

Denoting $\mu_u(\alpha) = E(\tilde{\delta}_u(\alpha))$ and $\sigma_u^2(\alpha) = Var(\tilde{\delta}_u(\alpha))$, it is assumed that the owners of the company have two objectives:

- *maximization of the expected value* $\mu_u(\alpha)$ of the company return on equity;
- minimization of the risk as measured by $\sigma_u^2(\alpha)$,

an approach similar to Markowitz model.

Instead of taking $\sigma_u^2(\alpha)$ as the measure of the risk, we can take some other measures, for example, those mentioned in Section 5. Let us denote by $\rho_u(\alpha)$ the generic risk measure.

The insurer's problem is

$$(MPP): \min_{\alpha \in D} \left(\rho_u(\alpha), -\mu_u(\alpha)\right)$$

where the feasible set is $D = \left\{\alpha \in \mathbf{R}^m \middle| \alpha_i \in [0,1], i = \overline{1,m}\right\}$.

Remark 3.1. The total investment constraint of the Markowitz model $\sum_{i=1}^{m} \alpha_i = 1$ would be meaningless in this context, and therefore it was dropped out (Schnieper, 2000).

4. Scalarization methods applied on insurer portfolio optimization problem

Now we apply different types of scalarization for this problem. Unless otherwise specified, we consider the initial capital (equity) u given.

Remark 4.1. In what it follows, for proving the next results, we will use the same approach as other authors, for example, Engau and Wiecek (2005).

4.1. Weighted-sum scalarization

A portfolio by weighted-sum scalarization is the optimal solution of

$$(PWS_w) : \min_{\alpha \in D} \{w_1 \rho_u(\alpha) - w_2 \mu_u(\alpha)\},$$

where $w_1, w_2 \geqq 0$ are given weighting parameter. As we mentioned in Section 2, $S = D$, $T = Y$, $U = \Phi$ and $\Pi = \mathbf{R}^2_{\geqq}$

We have the following result.

Proposition 4.1.1. Given the (MPP),

(i) If $\hat{\alpha} \in D$ is strictly optimal for the (PWS_w), then $\hat{\alpha}$ is efficient for the (MPP).

(ii) If $\hat{\alpha} \in D$ is optimal for the (PWS_w), then $\hat{\alpha}$ is weakly efficient for the (MPP).

(iii) If $\hat{\alpha} \in D$ is optimal for the (PWS_w) with $w \in \mathbf{R}^2_{>}$, then $\hat{\alpha}$ is efficient for the (MPP).

Example 4.1.1.[1] Let consider now that $\rho_u(\alpha) = \sigma^2_\alpha = Var(\delta_u(\alpha))$. Since $\tilde{\delta}_u(\alpha) = \sum_{i=1}^{m} \alpha_i \dfrac{E(\tilde{X}_i) + \ell_i - \tilde{X}_i}{u}$, we get $\mu_u(\alpha) = E(\tilde{\delta}_u(\alpha)) = \dfrac{\sum_{i=1}^{m} \alpha_i \ell_i}{u} = \dfrac{R(\underline{\alpha})}{u}$

and

$$\sigma^2_u(\alpha) = Var(\tilde{\delta}_u(\alpha)) = \frac{\sum_{i=1}^{m} \alpha_i \alpha_j \sigma^2_{ij}}{u^2} = \frac{V(\underline{\alpha})}{u^2},$$

where $\sigma_{ij} = \operatorname{cov}(\tilde{X}_i, \tilde{X}_j)$, and therefore $\sigma_u(\alpha) = \dfrac{[V(\underline{\alpha})]^{\frac{1}{2}}}{u}$.

Next, we choose $w_1 = 1 - \gamma$ and $w_2 = 2\gamma$, where $\gamma \in [0,1]$.

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[1] *We use the notations from Schnieper (2000).*

Remark 4.1.1. For $\gamma = 0$ we obtain the minimum variance-model, for $\gamma = 1$, we get the maximum mean-model and for $\gamma \in (0,1)$, we obtain the model from Markowitz (1952) or Schneiper (2000).

Now we let u to vary and propose the following optimization process in 3 steps, similar to the one proposed by Schnieper (2000):

1. Maximize the following function:

$$\max_{\alpha_1,\ldots,\alpha_m \in [0,1]} \gamma \sum_{i=1}^{m} \alpha_i \ell_i - (1-\gamma) \sum_{i,j=1}^{m} \alpha_i \alpha_j \sigma_{ij}^2$$

or maximize the risk return ratio (Sharpe's ratio)

$$\max_{\alpha_1,\ldots,\alpha_m \in [0,1]} \frac{R(\underline{\alpha})}{[V(\underline{\alpha})]^{\frac{1}{2}}}.$$

Remark 4.1.2. (Schnieper, 2000) The models simplify if the risks are uncorrelated.

In general, the above ratio is maximized for a whole set of admissible values of α. Let β_1 be the set of those values.

2. The second step requires the maximization of the net expected profit:

$$\max_{\underline{\alpha} \in \beta_1} \sum_{i=1}^{m} \alpha_i \ell_i .$$

Let $\underline{\alpha}_M$ denote the net retentions for which the above requirement is satisfied. Let $R = R(\underline{\alpha}_M)$ and $V = V(\underline{\alpha}_M)$.

3. The optimal amount of equity is defined by the solution of

$$\max_u \left[2\gamma \frac{1}{u} \sum_{i=1}^{m} \alpha_i \ell_i - (1-\gamma) \frac{1}{u^2} \sum_{i,j=1}^{m} \alpha_i \alpha_j \sigma_{ij}^2 \right].$$

The optimal amount of equity is thus

$$u = \frac{1-\gamma}{\gamma} \frac{V}{R}.$$

Example 4.1.2. Suppose again that $\rho_u(\alpha) = \sigma_\alpha^2 = Var(\delta_u(\alpha))$, $w_1 = 1-\gamma$ and $w_2 = 2\gamma$, where $\gamma \in (0,1)$. As in Schnieper (2000), we assume that there are two uncorrelated risks with expected profit ℓ_1 and ℓ_2 respectively, and standard deviation σ_1 and σ_2. We have

$$\mu_u(\alpha) = E\left(\tilde{\delta}_u(\alpha)\right) = E\left(\frac{\sum_{i=1}^{2}\alpha_i\left(E\left(\tilde{X}_i\right) + \ell_i - \tilde{X}_i\right)}{u}\right) = \frac{1}{u}\sum_{i=1}^{2}\alpha_i\ell_i = \sum_{i=1}^{2}\alpha_i\lambda_i ,$$

where $\lambda_i = \frac{\ell_i}{u}$, and

$$\sigma_u^2(\alpha) = Var\left(\tilde{\delta}_u(\alpha)\right) = Var\left(\frac{\sum_{i=1}^{2}\alpha_i\left(E\left(\tilde{X}_i\right) + \ell_i - \tilde{X}_i\right)}{u}\right) = \frac{1}{u^2}\sum_{i=1}^{2}\alpha_i^2\sigma_i^2 = \sum_{i=1}^{2}\alpha_i^2\tau_i^2$$

where $\tau_i = \frac{\sigma_i}{u}$.

The objective is

$$\max_{\alpha_i \in [0,1], i \in \overline{1,2}}\left[2\gamma\mu_\alpha(u) - (1-\gamma)\sigma_\alpha^2(u)\right],$$

which leads to the unconstrained optimum

$$\alpha_i = \frac{\gamma}{1-\gamma}\frac{\lambda_i}{\tau_i^2}.$$

Without any loss of generality we assume $\frac{\lambda_1}{\tau_1^2} \geqq \frac{\lambda_2}{\tau_2^2}$, and we make the following case distinction (Schnieper, 2000):

a) $\frac{\gamma}{1-\gamma} \leqq \frac{\tau_1^2}{\lambda_1}$ (note that $\frac{\gamma}{1-\gamma} \in (0,1]$ if $\gamma \in \left(0, \frac{1}{2}\right]$).

Remark 4.1.3. We could consider that $\frac{\gamma}{1-\gamma} = \tau$ - the *risk tolerance*.

In this case, $\frac{\gamma}{1-\gamma}\frac{\lambda_i}{\tau_i^2} \leqq 1$ for $i \in \overline{1,2}$, and then α_i's are defined as above, that is $\alpha_i = \frac{\gamma}{1-\gamma}\frac{\lambda_i}{\tau_i^2}$. Moreover,

$$\mu_\alpha = \frac{\gamma}{1-\gamma}\left(\frac{\lambda_1^2}{\tau_1^2} + \frac{\lambda_2^2}{\tau_2^2}\right)$$

and

$$\sigma_\alpha^2 = \left(\frac{\gamma}{1-\gamma}\right)^2 \left(\frac{\lambda_1^2}{\tau_1^2} + \frac{\lambda_2^2}{\tau_2^2}\right).$$

Here, $(\mu_\alpha, \sigma_\alpha)$ describes a straight line as τ varies.

b)
$$\frac{\tau_1^2}{\lambda_1} \leqq \frac{\gamma}{1-\gamma} \leqq \frac{\tau_2^2}{\lambda_2}.$$

In this case, $\frac{\gamma}{1-\gamma}\frac{\lambda_1}{\tau_1^2} \geqq 1$ and $\frac{\gamma}{1-\gamma}\frac{\lambda_2}{\tau_2^2} \leqq 1$, therefore $\alpha_1 = 1$ and $\alpha_2 = \frac{\gamma}{1-\gamma}\frac{\lambda_2}{\tau_2^2}$. Moreover,

$$\mu_\alpha = \lambda_1 + \frac{\gamma}{1-\gamma}\frac{\lambda_2^2}{\tau_2^2}$$

and

$$\sigma_\alpha^2 = \tau_1^2 + \left(\frac{\gamma}{1-\gamma}\right)^2 \frac{\lambda_2^2}{\tau_2^2}.$$

Here, $(\mu_\alpha, \sigma_\alpha)$ describes a hyperbole as τ varies.

c)
$$\frac{\gamma}{1-\gamma} \geqq \frac{\tau_2^2}{\lambda_2}.$$

In this case, $\frac{\gamma}{1-\gamma}\frac{\lambda_1}{\tau_1^2} \geqq 1$ and $\frac{\gamma}{1-\gamma}\frac{\lambda_2}{\tau_2^2} \geqq 1$, therefore $\alpha_1 = 1$ and $\alpha_2 = 1$. Moreover,

$$\mu_\alpha = \lambda_1 + \lambda_2$$

and

$$\sigma_\alpha^2 = \tau_1^2 + \tau_2^2,$$

which means that the efficient frontier degenerates into a single point.

4.2. Constrained-objective scalarization

A portfolio by constrained-objective scalarization is the optimal solution of

$$(PCO_\delta): \min_{\alpha \in D} \{\rho_u(\alpha) | \mu_u(\alpha) \geqq \delta\},$$

where $\delta \in \mathbf{R}$ being a given lower bound for $\mu_u(\alpha)$.

The relationships between optimal solutions of this scalarization and efficient decisions of the (MPP) are given below.

Proposition 4.2.1. Given the (MPP),

(i) If $\hat{\alpha} \in D$ is strictly optimal for the (PCO_δ), then $\hat{\alpha}$ is efficient for the (MPP).

(ii) If $\hat{\alpha} \in D$ is optimal for the (PCO_δ), then $\hat{\alpha}$ is weakly efficient for the (MPP).

Example 4.2.1 (Example 4.1.1. continued) Considering the notations and conditions from Example 4.1.1, the problem that have to be solved in this case is the following:

$$\begin{array}{l} \min\limits_{\alpha_1,\alpha_2 \in [0,1]} \left(\alpha_1^2 \tau_1^2 + \alpha_2^2 \tau_2^2\right) \\ \text{s.t. } \alpha_1 \lambda_1 + \alpha_2 \lambda_2 \geqq \delta \end{array}.$$

An alternative model with respect to constrained objective scalarization is the following (minimization is considered for the second component of objective function:

$$(PCO'_{\delta'}) : \max_{\alpha \in D} \left\{ \mu_u(\alpha) \middle| \rho_u(\alpha) \leqq \delta' \right\},$$

where $\delta' \in \mathbf{R}$ being a given upper bound for the risk measure.

The relationships between optimal solutions of this scalarization and efficient decisions of the (MPP) become:

Proposition 4.2.2. Given the (MPP),

(i) If $\hat{\alpha} \in D$ is strictly optimal for the $(PCO'_{\delta'})$, then $\hat{\alpha}$ is efficient for the (MPP).

(ii) If $\hat{\alpha} \in D$ is optimal for the $(PCO'_{\delta'})$, then $\hat{\alpha}$ is weakly efficient for the (MPP).

Example 4.2.2. (Example 4.1.1. continued) Considering the previous example, the problem to be solved now is

$$\begin{array}{l} \max\limits_{\alpha_1,\alpha_2 \in [0,1]} \left[\alpha_1 \lambda_1 + \alpha_2 \lambda_2\right] \\ \text{s.t. } \alpha_1^2 \tau_1^2 + \alpha_2^2 \tau_2^2 \leqq \delta \end{array}.$$

4.3. Guddat scalarization

We consider now the following problem

$$\left(PG_{\alpha^0,w}\right) : \min_{\alpha \in D} \left\{ w_1 \rho_u(\alpha) - w_2 \mu_u(\alpha) \middle| \rho_u(\alpha) \leqq \rho_u(\alpha^0), \mu_u(\alpha) \geqq \mu_u(\alpha^0) \right\},$$

where $\alpha^o \in D$ is a given feasible portfolio and $w_1, w_2 \geqq 0$ are given weighting parameters.

Denote $y^o = \left(\rho_u(\alpha^o), -\mu_u(\alpha^0)\right)$ and observe that for this scalarization $T = \left\{ y \in Y \middle| y \leqq y^o \right\}$, $U = \Phi$ and $\Pi = D \times \mathbf{R}^2_{\geqq}$.

Now we have the following result.

Proposition 4.3.1. Given the (MPP),

(i) If $\hat{\alpha} \in D$ is strictly optimal for the $\left(PG_{\alpha^0,w}\right)$, then $\hat{\alpha}$ is efficient for the (MPP).

(ii) If $\hat{\alpha} \in D$ is optimal for the $\left(PG_{\alpha^0,w}\right)$, then $\hat{\alpha}$ is weakly efficient for the (MPP).

(iii) If $\hat{\alpha} \in D$ is optimal for the $\left(PG_{\alpha^0,w}\right)$ with $w \in \mathbf{R}^m_>$, then $\hat{\alpha}$ is efficient for the (MPP).

Example 4.3.1. (Example 4.1.1. continued) In this case, the problem that has to be solved is

$$\min_{\alpha_1,\alpha_2\in[0,1]}\left[w_1\left(\alpha_1^2\tau_1^2+\alpha_2^2\tau_2^2\right)-w_2\left(\alpha_1\lambda_1+\alpha_2\lambda_2\right)\right]$$
$$\text{s.t. } \alpha_1^2\tau_1^2+\alpha_2^2\tau_2^2 \leqq \left(\alpha_1^0\right)^2\tau_1^2+\left(\alpha_2^0\right)^2\tau_2^2 \quad .$$
$$\alpha_1\lambda_1+\alpha_2\lambda_2 \geqq \alpha_1^0\lambda_1+\alpha_2^0\lambda_2$$

4.4. Benson scalarization

We consider the following problem which leads to the Benson optimal portfolio

$$\left(P\widetilde{B}_{\alpha^0}\right):\ \min_{\alpha\in D}\left\{\rho_u(\alpha)-\mu_u(\alpha)\middle|\rho_u(\alpha)\leqq\rho_u\left(\alpha^o\right),\ \mu_u(\alpha)\geqq\mu_u\left(\alpha^o\right)\right\},$$

where $\alpha^o \in D$ is a given feasible portfolio.

Note that $\left(P\widetilde{B}_{\alpha^0}\right)=\left(PG_{\alpha^0,1}\right)$. Therefore, we can immediately derive the next result.

Proposition 4.4.1. Given the (MPP), if $\hat{\alpha} \in D$ is optimal for the $\left(P\widetilde{B}_{\alpha^0}\right)$, then $\hat{\alpha}$ is efficient for the (MPP).

Example 4.4.1. (Example 4.1.1. continued) In this case, the problem that has to be solved is

$$\min_{\alpha_1,\alpha_2\in[0,1]}\left[\left(\alpha_1^2\tau_1^2+\alpha_2^2\tau_2^2\right)-\left(\alpha_1\lambda_1+\alpha_2\lambda_2\right)\right]$$
$$\text{s.t. } \alpha_1^2\tau_1^2+\alpha_2^2\tau_2^2 \leqq \left(\alpha_1^0\right)^2\tau_1^2+\left(\alpha_2^0\right)^2\tau_2^2 \quad .$$
$$\alpha_1\lambda_1+\alpha_2\lambda_2 \geqq \alpha_1^0\lambda_1+\alpha_2^0\lambda_2$$

The problem simplifies if we know that between α_1 and α_2 there is some linear (for example) relation, such $\alpha_1+\alpha_2=ct$.

4.5. Min-max scalarization

A min-max portfolio is the optimal solution of

$$(PMM):\ \min_{\alpha\in D}\max\left(\rho_u(\alpha),-\mu_u(\alpha)\right).$$

Observe that for this scalarization $S=D$, $T=Y$, $U=\Pi=\Phi$. We get the following properties.

Note in this case that $\rho_u(\alpha) \geqq -\mu_u(\alpha)$, and therefore the problem becomes

$$(PMM): \min_{\alpha \in D} \rho_u(\alpha),$$

the minimum-variance model.

Example 4.5.1. (Example 4.1.1. continued) Then the problem to be solved is

$$\min_{\alpha_1, \alpha_2 \in [0,1]} \left(\alpha_1^2 \tau_1^2 + \alpha_2^2 \tau_2^2\right).$$

4.6. Tchebycheff-norm scalarization

A Tchebycheff-norm portfolio is the optimal solution of

$$(PTN_{r,w}): \min_{\alpha \in D} \max\{w_1(\rho_u(\alpha) - r_1), w_2(-\mu_u(\alpha) - r_2)\},$$

where $r \in \mathbf{R}^2$ is a given reference or utopia point (Steuer (1986)) and $w \in \mathbf{R}^2_{\geqq}$ is a given weighting parameter.

Remark 4.6.1. The min-max scalarization discussed previously is a special case of the weighted Tchebycheff-norm formulation. More precisely, if we choose the reference point $r = (0,0)^T \in \mathbf{R}^2$ and the weighting parameter $w = (1,1)^T \in \mathbf{R}^2_{>}$, then $(PMM) = (PTN_{0,1})$.

Now, we have the following results:

Proposition 4.6.1. Given the (MPP),

(i) If $\hat{\alpha} \in D$ is strictly optimal for the $(PTN_{r,w})$, then $\hat{\alpha}$ is efficient for the (MPP).

(ii) If $\hat{\alpha} \in D$ is optimal for the $(PTN_{r,w})$, then $\hat{\alpha}$ is weakly efficient for the (MPP).

Example 4.6.1. (Example 4.1.1. continued) Denote by

$$\varphi(\alpha_1, \alpha_2) = \begin{cases} w_1\left(\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 - r_1\right), \text{ if } w_1\left(\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 - r_1\right) \geqq w_2\left(\lambda_1\alpha_1 + \lambda_2\alpha_2 - r_2\right) \\ w_2\left(\lambda_1\alpha_1 + \lambda_2\alpha_2 - r_2\right), \text{ if } w_1\left(\alpha_1^2\sigma_1^2 + \alpha_2^2\sigma_2^2 - r_1\right) < w_2\left(\lambda_1\alpha_1 + \lambda_2\alpha_2 - r_2\right) \end{cases}.$$

Then solve $\min_{\alpha_1, \alpha_2 \in [0,1]} \varphi(\alpha_1, \alpha_2)$.

4.7. Gasimov scalarization

We introduce the Gasimov portfolio as the optimal solution of the following problem

$$(GPP_{a,w}): \min_{\alpha \in D}\left\{a\left(\left|\rho_u(\alpha) - a_1\right| + \left|-\mu_u(\alpha) + a_2\right|\right) + \left(w_1(\rho_u(\alpha) - a_1) + w_2(-\mu_u(\alpha) + a_2)\right)\right\}.$$

Example 4.7.1. (Example 4.1.1. continued) In this case, the problem to be solved is

$$\min_{\alpha_1,\alpha_2\in[0,1]} a\left[\left|\alpha_1^2\tau_1^2+\alpha_2^2\tau_2^2-a_1\right|+\left|a_2-\alpha_1\lambda_1-\alpha_2\lambda_2\right|\right]+w_1\left(\alpha_1^2\tau_1^2+\alpha_2^2\tau_2^2-a_1\right)+$$
$$+w_2\left(a_2-\alpha_1\lambda_1-\alpha_2\lambda_2\right)$$

Before presenting some results relative to $(GPP_{a,w})$, we will remind certain definitions.

Definition 4.7.1. Let Y be a nonempty set of $\mathbf{R}^q$.

(i) An element $y \in Y$ is called **non-dominated** if $\left(\{y\}-\mathbf{R}^q_{\geqq}\right)\cap Y=\{y\}$, i.e. there is no other $y' \in Y$ such that $y' \le y$.

(ii) An element $y \in Y$ is called **properly non-dominated** (in the sense of Benson) if y is a non-dominated element of Y and the zero element of $\mathbf{R}^q$ is a non-dominated element of $\operatorname{cl} cone\left(Y+\mathbf{R}^q_{\geqq}-y\right)$, where $\operatorname{cl} A$ denotes the closure of a set A and $coneA=\{\theta a \mid \theta \ge 0, a \in A\}$.

Definition 4.7.2. Consider now the problem (VPP) given in Section 2. A feasible solution $\hat{\alpha} \in D$ is called *(properly) Benson efficient* if $\hat{y} = f(\hat{\alpha})$ is a (properly) non-dominated element of Y, or, alternatively, $\hat{\alpha} \in D$ is said to be *properly efficient solution of* (VPP) *in the sense of Benson* if $\operatorname{cl} cone\left(f(D)+\mathbf{R}^q_{\geqq}-f(\hat{\alpha})\right)\cap\left(-\mathbf{R}^q_{\geqq}\right)=\{0\}$.

We go back again to the $(GPP_{a,w})$ problem. We get the following results concerning Gasimov portfolios, extending in some ways the results of Gasimov (2001).

Proposition 4.7.1. A feasible solution $\hat{\alpha} \in D$ is Benson proper efficient solution if and only if there exist $a, a_1, a_2 \in \mathbf{R}$ and $w \in \mathbf{R}^2_{\geqq}$, with $0 \le a \le \min\{w_1, w_2\}$, such that $\hat{\alpha}$ is an optimal solution to $(GPP_{a,w})$.

Theorem 4.7.1. Suppose that for some (a,w), $a \in \mathbf{R}$ and $w \in \mathbf{R}^2_{\geqq}$, with $0 \le a \le \min\{w_1, w_2\}$, a feasible solution is an optimal solution to the scalar minimization problem

$$\min_{\alpha\in D}\left\{a\left(\left|\rho_u(\alpha)\right|+\left|-\mu_u(\alpha)\right|\right)+\left(w_1\rho_u(\alpha)-w_2\mu_u(\alpha)\right)\right\},$$

then $\hat{\alpha}$ is a Benson proper efficient solution to (MPP).

Example 4.7.1. (Example 4.1.1. continued) In the example considered, the problem to be solved is

$$\min_{\alpha_1,\alpha_2\in[0,1]} a\left[\alpha_1^2\tau_1^2+\alpha_2^2\tau_2^2+\alpha_1\lambda_1+\alpha_2\lambda_2\right]+w_1\left(\alpha_1^2\tau_1^2+\alpha_2^2\tau_2^2\right)-w_2\left(\alpha_1\lambda_1+\alpha_2\lambda_2\right).$$

Theorem 4.7.2. Let $\hat{\alpha}\in D$ be a Benson proper efficient solution to (MPP). Then there exists a vector (a,w) with $a\in\mathbf{R}$, $w\in\mathbf{R}^2_{\geqq}$ and $0\le a\le\min_{i=1,2} w_i$, such that $\hat{\alpha}$ is an optimal solution to the scalar minimization problem $(GPP_{a,w})$

$$\min\left\{a\left[\left(\left|\rho_u(\alpha)\right|-\left|\rho_u(\hat{\alpha})\right|\right)+\left(\left|\mu_u(\alpha)\right|-\left|\mu_u(\hat{\alpha})\right|\right)\right]+\left[w_1\left(\rho_u(\alpha)-\rho_u(\hat{\alpha})\right)+w_2\left(\mu_u(\alpha)-\mu_u(\hat{\alpha})\right)\right]\right\}.$$

5. Some final remarks

In what concerns the insurer portfolio optimization problem (MPP), instead of taking the mean and the variance as the measures of the risk, we could consider other measures, which have to be either minimized or maximized. For example, other risk measures that could be taken into account are:

- *Value-at-Risk (VaR) at level p* - $VaR_p(X)=Q_p(X)=\inf\left\{x\in\mathbf{R}\middle|F_X(x)\ge p\right\}$, where $F_X(x)=\Pr(X\le x)$.
- *Tail Value-at-Risk at level p* - $TVaR_p(X)=\frac{1}{1-p}\int_p^1 Q_q(X)dq$. In fact, it is the arithmetic average of the quantiles of X, from p on.
- *Conditional Tail Expectation at level p* - $CTE_p(X)=E\left(X\middle|X>VaR_p(X)\right)$, where $p\in(0,1)$. Loosely speaking, the conditional tail expectation at level p is equal to the mean of the top $(1-p)\%$ losses. It can also be interpreted as the VaR at level p augmented by the average exceedance of the claims X over that quantile, given that such exceedance occurs.
- *The Expected Shortfall at level p* - $ESF_p(X)=E\left(\left(X-VaR_p(X)\right)_+\right)$, where $p\in(0,1)$.

We conclude this paper by reminding that various authors have proposed different models based on the Markowitz optimization problem. For example, Arthur and Ghandforoush (1987) considered some objective and subjective measures for assets leading to a simple linear programming model. Konno (1990) constructed a piecewise risk function to replace the covariance, which led also to a linear programming model. Markowitz et al. (1994) proposed a method which avoids actual computation of the covariance matrix and Morita et al (1989) applied stochastic linear knapsack model to the portfolio selection model. Furthermore, Ballestero and Romero (1996) were the first who proposed a compromise programming model for an "average" investor, which was modified to approximate the optimum portfolio for an individual investor (Ballestero (1998)). Hallerbach and Spronk (1997) explained that most models do not incorporate the multidimensional nature of the problem and outline a framework for such a view on portfolio management. In 2004, Ehrgott et al presented an objective hierarchy and formulated a multicriteria optimization model which uses five objective functions. Steuer et al. (2005) derived a suitable portfolio investor problem, taking into account objectives other than expected return and variance in their portfolio selection problem.

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