9 POLYNOMIAL INTERPOLATION AND APPLICATIONS TO AUTOREGRESSIVE MODELS

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Abstract

Polynomial interpolation can be used to approximate functions and their derivatives. Some autoregressive models can be stated by using polynomial interpolation and function approximation.

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The main issue

The major goal of this article is to state a general method in order to create forecast models, which can be used in time series approximation. This method is based on some fundamental mathematical formulas, such as polynomial interpolation and function approximation.

In a time series, such as the exchange rate on daily or monthly basis, it is not possible to use real functions $f : \mathrm{R} \rightarrow \mathrm{R}$ because the domain of definition, consisting of real numbers, cannot be associated with fixed steps, such as days or months.

As it is known, any mathematical tool for function approximation uses derivatives. But, for a time series, the differential quotient $\frac{y_{i+1} - y_i}{x_{i+1} - x_i}$ cannot be used if the denominator is not a real number!

Furthermore, we will prove that it is possible to use derivatives, even for time series, by expressing derivatives approximation through function values, instead of differential quotients. More accurately, we will see that a derivative, corresponding to a value y_t may be expressed by using only precedent values of the time series: y_{t-1}, y_{t-2}, etc.

Finally, a Taylor polynomial approximation may be used in a time series forecasting, including derivatives, by using only precedent values. This technique evolves into a

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general method, in order to create autoregressive models, as it will be stated in a dedicated section of this article. These models can be used in forecasting, as this paper will show it at the end.

To begin with, let us consider the following classical framework.

Given the set (x_1, y_1), (x_1, y_1), ... (x_1, y_1) in R^2, $x_i \neq x_j, i \neq j$, find a polynomial which satisfies the condition:

$P(x_i) = y_i, \forall i = \overline{1,m}$

Apparently, it is possible to use a polynomial function, in any cases in which we have a set of statistical (or experimental) data, but we don't know the exact function which denote the dependence between "*x*"-points and "*y*"-points.

In fact, in most cases, the interpolation polynomial is not used "*as it is*", being a very useful tool, in order to approximate some other operations, such as derivation or integration.

The main result which underline our issue is the following:

***Theorem* 1.** There exists an unique polynomial:

$$P(x) = a_0 x^{m-1} + a_1 x^{m-2} + ... + a_{m-2} x + a_{m-1},$$

having the degree $m-1$, which satisfies the condition (1).

Demonstration. The interpolation conditions are:

$$\begin{cases} a_0 x_1^{m-1} + a_1 x_1^{m-2} + ... + a_{m-2} x_1 + a_{m-1} = y_1 \\ a_0 x_2^{m-1} + a_1 x_2^{m-2} + ... + a_{m-2} x_2 + a_{m-1} = y_2 \\ .... \\ a_0 x_m^{m-1} + a_1 x_m^{m-2} + ... + a_{m-2} x_m + a_{m-1} = y_m \end{cases}$$

where the unknowns are $a_0, a_1, ... a_{m-1}$. While the above system is ***Vandermonde***, it results that there exists an unique solution.

***Lagrange's* formula**

It is quite easy to observe that the above theorem is „*time and space*" expensive, in order to determine the polynomial. This is the reason for introducing some other methods.

Lagrange's formula is based on the following family of particular polynomials:

$$\Phi_i(x) = \frac{(x - x_1)...(x - x_{i-1})(x - x_{i+1})...(x - x_m)}{(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_m)}, i = \overline{1,m}$$

By definition, the ***Lagrange*** (interpolation) polynomial is:

$$L(x) = \Phi_1(x)y_1 + \Phi_2(x)y_2 + ... + \Phi_m(x)y_m$$

Theorem* 2**. The ***Lagrange's polynomial coincides with the interpolation polynomial.

Demonstration. Indeed, the ***Lagrange's*** polynomial has the degree $m-1$ and it is easy to observe that the interpolation condition (1) is fulfilled:

$$L(x_i) = \Phi_1(x_i)y_1 + \Phi_2(x_i)y_2 + ... + \Phi_m(x_i)y_m = \Phi_i(x_i)y_i = y_i$$

Obviosly, it is quite easy to calculate the above polynom.

Interpolating an unknown function

In many cases, the set of interpolated values belongs to the graph of an unknown function, i.e. $y_i = f(x_i), i = \overline{1,m}$ and the problem is:

Find the polynomial *P*, which has the (maximum) degree $m-1$, and satisfies:

$$P(x_i) = f(x_i), i = \overline{1,m}$$

Taking into account the previous results, such a polynomial exists and it is unique.

The case of the function interpolation has various applications, for example, in approximate derivation and integration of unknown functions. These applications are based on the following theorem, which explains the magnitude of the error made by approximating the function *f*, by the polynomial *P*.

***Theorem* 3.** Let us consider the function $f:[a,b] \to \mathrm{R}$, $f \in C^m([a,b])$ and the set of values $x_1, x_2, ..., x_m \in [a,b], x_i \neq x_j, \forall i \neq j$, then, the interpolating polynomial *P*, satisfies:

$$|f(x) - P(x)| \leq \frac{1}{m!} \sup_{u \in [a,b]} |f^{(m)}(u)| |(x-x_1)(x-x_2)...(x-x_m)|$$

Demonstration. We will use the auxiliary function:

$$\varphi:[a,b] \to \mathrm{R}, \varphi(t) = f(t) - P(t) - a(t-x_1)...(t-x_m)$$

where *a* is defined by the condition $\varphi(x) = 0, x \neq x_i, i = \overline{1,m}$, which implies:

$$a = \frac{f(x) - P(x)}{(x-x_1)...(x-x_m)} \qquad (2)$$

On the other hand, $\varphi(x) = \varphi(x_1) = ... = \varphi(x_m) = 0$, thus we may use the ***Rolle's*** theorem *m*-times, corresponding to each pair of above values. There exists *m* different values, so that the derivative φ' has the value 0.

The above assertion may be used recurrently, for the successive derivatives of function φ, and finally it results that there exists $\xi \in (a,b)$ so that:

$$\varphi^{(m)}(\xi) = 0$$

But,

$$\varphi^{(m)}(t) = f^{(m)}(t) - P^{(m)}(t) - \left(a(t - x_1)...(t - x_m)\right)^{(m)} = f^{(m)}(t) - 0 - m!a$$

which shows that $\varphi^{(m)}(\xi) = f^{(m)}(\xi) - m!a = 0$

The last result gives a new for *a*, i.e.:

$$a = \frac{f^{(m)}(\xi)}{m!}$$

and by using (2) it results:

$$\frac{f(x) - P(x)}{(x - x_1)...(x - x_m)} = \frac{f^{(m)}(\xi)}{m!}$$

In conclusion, there exists $\xi \in (a,b)$, which depends on $x, x_1,...,x_m$, so that:

$$f(x) - P(x) = \frac{f^{(m)}(\xi)}{m!}(x - x_1)...(x - x_m)$$

and the conclusion is obvious, by using the modulus:

$$|f(x) - P(x)| \le \frac{1}{m!} \sup_{u \in [a,b]} |f^{(m)}(u)| |x - x_1| |x - x_2| |x - x_m|$$

Examples:

a) One point interpolation, (*c*,*f*(*c*)).

$$P(x)=f(c)$$
$$|f(x) - P(x)| \le \sup_{u \in [a,b]} |f^{'}(u)| |x - c|$$

b) Two points interpolation, (*a*,*f*(*a*)), (*b*,*f*(*b*)).

$$P(x) = \frac{x - b}{a - b} f(a) + \frac{x - a}{b - a} f(b)$$
$$|f(x) - P(x)| \le \frac{1}{2} \sup_{u \in [a,b]} |f^{''}(u)| |(x - a)(x - b)|$$

Newton's formula, based on divided differences

Let us consider the function $f:[a,b] \to \mathbb{R}$ and $x_1, x_2,...,x_m \in [a,b]$, $x_i \ne x_j, \forall i \ne j$. In what follows, we will denote by $g(x; x_1, x_2,..., x_m)$ the interpolation polynomial, corresponding to the set of values $(x_1, f(x_1)), (x_2, f(x_2)),...,(x_m, f(x_m))$.

***Definition* 1**. By definition, the divided difference of function *f*, with respect the points $x_1, x_2,..., x_m$, is the coefficient of x^{m-1}, in $g(x; x_1, x_2,..., x_m)$.

We will denote this value by $f[x_1, x_2,..., x_m]$

***Observation* 1.** $f\left[x_1,x_2,...,x_m\right]=f\left[x_{i1},x_{i2},...,x_{im}\right]$ for any permutation $\left(i1,i2,...,im\right)$

***Observation* 2.** Taking into account the two precedent examples, we have:

$$f\left[c\right]=f\left(c\right)$$

$$f\left[a,b\right]=\frac{f\left(b\right)-f\left(a\right)}{b-a}$$

***Theorem* 4 (*Newton's* formula).** Let *f* be the function

$$f:\left[a,b\right]\to \mathrm{R} \text{ and } x_1,x_2,\ldots,x_m\in\left[a,b\right], x_i\neq x_j, \forall i\neq j$$

Then

$$g(x;x_1,x_2,...,x_m)=f[x_1]+f[x_1,x_2](x-x_1)+...$$
$$+f[x_1,x_2,...,x_m](x-x_1)(x-x_2)...(x-x_{m-1})$$

Demonstration. First of all, we will consider the polynomial:

$$k\left(x\right)=g(x;x_1,x_2,...,x_m)-g(x;x_1,x_2,...,x_{m-1})$$

which has the degree $m-1$ and $m-1$ zeros, i.e. $x_1,x_2,...,x_{m-1}$; consequently:

$$k\left(x\right)=f\left[x_1,x_2,...,x_m\right]\left(x-x_1\right)...\left(x-x_{m-1}\right)$$

because the power x^{m-1} can be found only in $g(x;x_1,x_2,...,x_m)$.

By comparing the two formulas, it results:

$$g(x;x_1,x_2,...,x_m)=g(x;x_1,x_2,...,x_{m-1})$$
$$+f[x_1,x_2,...,x_m](x-x_1)(x-x_2)...(x-x_{m-1}) \qquad (3)$$

The above formula can be used recursively:

$$g(x;x_1)=f[x_1] \text{ (see first example)}$$
$$g(x;x_1,x_2)=g(x;x_1)+f[x_1,x_2](x-x_1)$$
$$g(x;x_1,x_2,...,x_m)=g(x;x_1,x_2,...,x_{m-1})$$
$$+f[x_1,x_2,...,x_m](x-x_1)(x-x_2)...(x-x_{m-1})$$

and, by summing up, we obtain the ***Newton's*** formula.

Aitken's formula

Another way to express interpolation polynomial is based on ***Aitken's*** formula, as it is stated by the following theorem.

Theorem 5 (Aitken)

$$g(x;x_1,x_2,...,x_m)=\frac{x-x_m}{x_1-x_m}g(x;x_1,x_2,...,x_{m-1})+\frac{x-x_1}{x_m-x_1}g(x;x_2,x_3,...,x_m)$$

Demonstration. All we have to do is to observe that the right side of the above formula represents a polynomial and this polynomial satisfies all the interpolating conditions, i.e. it coincides with the values of function *f*, in each point x_i. But, the interpolation polynomial is unique, which proves the ***Aitken's*** formula.

Observation. Taking into account the coefficients of x^{m-1}, on both sides of ***Aitken's*** formula, it results the following recurrent formula of divided differences:

$$f[x_1,x_2,...,x_m]=\frac{f[x_1,x_2,...,x_{m-1}]-f[x_2,x_3,...,x_m]}{x_1-x_m}$$

How to calculate the interpolation polynomial, based on the *Newton's* formula

In what follows, we present a ***MathCAD*** based strategy, in order to calculate the interpolating polynomial, by using the ***Newton's*** formula.

First, we define the values to be interpolated, by using the table "f" and two vector selections, "x" and "y".

f :=

	0	1
0	-1	-1
1	-3	-29
2	-2	-9
3	3	31

$$x := f^{\langle 0\rangle}$$
$$y := f^{\langle 1\rangle}$$
$$m := 4$$

The divided differences are calculated in a recursive way as it is shows below, by keeping only those are used in the ***Newton's*** formula.

```
difdiv(x, y, m) := | for  j ∈ 2.. m
                   |   for  k ∈ j.. m
                   |     | i ← m + j − k
                   |     | y[i−1] ← (y[i−2] − y[i−1]) / (x[i−j] − x[i−1])
                   | return  y
```

The values of divided differences may be memorized in the "y" vector or into a new one.

$$y := \text{difdiv}(x, y, m)$$

Finally, the polynomial is implemented by using a Horner's like algorithm.

```
pol(x, y, u) := | s ← y_{m−1}
                | for  k ∈ 1.. m − 1
                |   | i ← m − k
                |   | s ← y_{i−1} + (u − x_{i−1})·s
                | return  s
```

Derivate approximation

***Theorem* 6**. Let *f* be a function in $C^{m-1}[a,b]$ and $x_1, x_2, ..., x_m \in [a,b]$ then there exists $\xi \in (a,b)$ such that $f[x_1, x_2, ..., x_m] = \frac{f^{(m-1)}(\xi)}{(m-1)!}$

Demonstration. Les us consider $\psi(x) = f(x) - g(x; x_1, x_2, ..., x_m)$. Using the some technique as in the ***theorem*** 3, we can observe that ψ has *m* different zeros, namely $x_1, x_2, ..., x_m$. Consequently, by using the ***Rolle*** theorem, there are $m-1$ different zeros for the derivative ψ'; it results there are $m-2$ different zeros for the second derivative ψ'', etc. Finally, there exist $\xi \in (a,b)$ such that $\psi^{(m-1)}(\xi) = 0$, i.e. $f^{(m-1)}(\xi) - g^{(m-1)}(\xi; x_1, x_2, ..., x_m) = 0$. But, by using ***Newton's*** formula, it results $g^{(m-1)}(\xi; x_1, x_2, ..., x_m) = (m-1)! f[x_1, x_2, ..., x_m]$, thus:

$$f^{(m-1)}(\xi) = (m-1)! f[x_1, x_2, ..., x_m]$$

***Definition* 2.** By definition, a multiple divided difference is:

$$f[x_1, x_2, ..., x_p, u, u] = \lim_{\varepsilon \to 0} f[x_1, x_2, ..., x_p, u+\varepsilon, u]$$

Remark. $\frac{d}{du} f[x_1, x_2, ..., x_p, u] = f[x_1, x_2, ..., x_p, u, u]$

In order to approximate the derivate of a function *f*, the main idea is to approximate it by the derivate of the interpolation polynomial.

Let us consider the following relation, which result from (3) by using interpolating points x_1, x_2, ..., x_m and *u*:

$g(x; x_1, x_2, ..., x_m, u) = g(x; x_1, x_2, ..., x_m) + f[x_1, x_2, ..., x_m, u](x - x_1)(x - x_2)...(x - x_m)$ For $x = u$, we obtain:

$$f(u) = g(u; x_1, x_2, ..., x_m) + f[x_1, x_2, ..., x_m, u]\pi(u)$$

where $\pi(u) = \prod_{i=\overline{1,m}} (u - x_i)$

It results, by derivation and by applying ***theorem* 6**:

$$f'(u) = g'(u; x_1, x_2, ..., x_m) + \frac{f^{(m+1)}(\xi_1)}{(m+1)!}\pi(u) + \frac{f^{(m)}(\xi_2)}{m!}\pi'(u)$$

In conclusion, when we approximate $f'(u)$ by $g'(u; x_1, x_2, ..., x_m)$, then the error is:

$$\frac{f^{(m+1)}(\xi_1)}{(m+1)!}\pi(u) + \frac{f^{(m)}(\xi_2)}{m!}\pi'(u)$$

Examples of derivate approximation

1.Two points, x_1 and x_2

$$\pi(u) = (u - x_1)(u - x_2)\,,\ \pi'(u) = 2u - x_1 - x_2$$

For $x_1 = u$ and $x_2 = u - h$ it results $f'(u) = \frac{f(u) - f(u-h)}{h} + \frac{h}{2}f^{(2)}(\xi_2)$, which means that:

$$f'(u) \approx \frac{f(u) - f(u-h)}{h}\text{, the error magnitude being } \frac{h}{2}f^{(2)}(\xi_2)$$

The above formula shows that for small values of *h*, we have a good approximation; say for $h = 10^{-3}$ the divided difference approximate the derivative with an error of tree decimal digits.

2.Three points, x_1, x_2 and x_3 **.**

$$\pi(u) = (u - x_1)(u - x_2)(u - x_3)\,,$$

$$\pi'(u) = \left(u - x_1\right)\left(u - x_2\right) + \left(u - x_1\right)\left(u - x_3\right) + \left(u - x_2\right)\left(u - x_3\right)$$

For $x_1 = u, x_2 = u - h, x_3 = u - 2h$:

$$f'(u) = \frac{3f(u) - 4f(u-h) + f(u-2h)}{2h} + \frac{h^2}{3}f^{(3)}(\xi_2)$$

3.Second derivate

$$f''(u) = g''(u; x_1, x_2, ..., x_m) + \frac{f^{(m+2)}(\xi_1)}{(m+2)!}\pi(u) + 2\frac{f^{(m+1)}(\xi_2)}{(m+1)!}\pi'(u) +$$

$$+ \frac{f^{(m)}(\xi_3)}{m!}\pi''(u)$$

$$g''(u; x_1, x_2, x_3) = 2f\left[x_1, x_2, x_3\right]$$

$$\pi''(u) = 2\left[\left(u - x_1\right) + \left(u - x_2\right) + \left(u - x_3\right)\right]$$

For $x_1 = u - h, x_2 = u, x_3 = u + h$:

$$g''(u; x_1, x_2, x_3) = 2f\left[x_1, x_2, x_3\right] = 2f\left[u - h, u, u + h\right]$$

$$= \frac{f\left(u - h\right) - 2f\left(u\right) + f\left(u + h\right)}{h^2}$$

$$\pi(u) = 0\,,\ \pi'(u) = -h^2\,,\ \pi''(u) = 0$$

$$f''(u) = \frac{f(u-h) - 2f(u) + f(u+h)}{h^2} - \frac{h^2}{24} f^{(4)}(\xi_2)$$

Autoregressive models

We shall present two ways, in order to define some autoregressive models, based on polynomial interpolation.

Use of *Lagrange's* polynomial

First, let us observe that if (y_t) is a time series, then the interpolation polynomial, based on values $y_{t-m},...,y_{t-1}$ may be used in order to estimate y_t. To do this it is enough to calculate ***Lagrange's*** polynomial, as we can show in the following examples.

1.Two points, formally denoted as (x_1, y_1) and (x_2, y_2), determines the ***Lagrange*** polynomial:

$$L(x) = y_1 \frac{x - x_2}{x_1 - x_2} + y_2 \frac{x - x_1}{x_2 - x_1}$$

Taking into account a constant step, $x_2 - x_1 = x_3 - x_2 = h$, we may calculate the value $y_3 = L(x_3) = y_1 \frac{x_3 - x_2}{x_1 - x_2} + y_2 \frac{x_3 - x_1}{x_2 - x_1} = -y_1 + 2y_2$ and the model is:

$$y_t = 2y_{t-1} - y_{t-2}$$

which corresponds to a linear model.

2.Three points, formally denoted as (x_1, y_1), (x_2, y_2) and (x_3, y_3), determines the ***Lagrange's*** polynomial:

$$L(x) = y_1 \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} + y_2 \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Taking into account a constant step, $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = h$, we may calculate $y_4 = L(x_4) = y_1 \frac{2h \cdot h}{(-h) \cdot (-2h)} + y_2 \frac{3h \cdot h}{h \cdot (-h)} + y_2 \frac{3h \cdot 2h}{2h \cdot h} = y_1 - 3y_2 + 3y_3$ and the model is:

$$y_t = 3y_{t-1} - 3y_{t-2} + y_{t-3}$$

which correspond to a second degree polynomial.

3. Four points, formally denoted as (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), determines, by using the some technique as above, the three degree model:

$$y_t = 4y_{t-1} - 6y_{t-2} + 4y_{t-3} - y_{t-4}$$

Use of *Taylor's* polynomial

There are many mathematical tools in order to approximate functions. One of the most important is ***Taylor's*** series associated to a function *f*. Let us suppose that the function $f:[a,b]\to \mathrm{R}$ is in $C^{\infty}[a,b]$, having all derivatives uniformly bounded, i.e. there exists $M>0$ so that $\left|f^{(n)}(x)\right|\le M,\forall n\in \mathrm{N},\forall x\in [a,b]$, then:

$$f(x+h)=f(x)+\frac{f'(x)}{1!}h+...+\frac{f^{(n)}(x)}{n!}h^n+...,\forall x,x+h\in[a,b],$$

and the approximate formula holds:

$$f(x+h)\approx f(x)+\frac{f'(x)}{1!}h+...+\frac{f^{(n)}(x)}{n!}h^n,$$

the error magnitude being at most $\frac{M}{(n+1)!}h^{n+1}$

In some models, it is quite possible that we don't know the derivatives of the function *f*. Thus, we have to approximate derivatives, the final approximation being depending on that.

As for example, for $n=2$, by using derivative approximation, it results:

$$f(u+h)\approx f(u)+\frac{3f(u)-4f(u-h)+f(u-2h)}{2}+\frac{f(u-h)-2f(u)+f(u+h)}{2}$$

$$f(u+h)\approx 3f(u)-3f(u-h)+f(u-2h)$$

The above formula reveals a very important fact, if we consider a time series. In such a case, even the values are real numbers, the definition domain cannot be associated with one interval of real numbers. But, as we observe, the formula contains only function's values, so, for a time series (y_t) it may be expressed as:

$$y_{t+1}\approx 3y_t-3y_{t-1}+y_{t-2}$$

which means that the future value, y_{t+1}, can be predicted, approximately as we already show, as depending on three precedents values.

It is quite remarkable that we have obtained the some autoregressive formula, by using the ***Taylor's*** polynomial, as we already stated, by using the ***Lagrange's*** polynomial !

Conclusions

In time series analysis, it is not possible to use models based on derivatives of real functions $f:\mathrm{R}\to\mathrm{R}$. For example, many statistical data are based on fixed periods of time, days, weeks, months, etc., which don't have the meaning of real numbers or values.

But, some fundamental mathematical approximation formulas, such as Taylor's or Lagrange's, use derivatives.

As we concluded in this paper, by using polynomial interpolation, it is possible to express a function or polynomial approximation, by using precedent values instead of argument values, i.e. by using some autoregressive formulas.

Example

We used data released by *National Bank of Romania*, representing the exchange rate EU/RON, in 2007, between January 15th and November 9th. Taking into account each four day group of values, we calculated the "*next*", or "*future*" value for the exchange rate, by using the four point autoregressive model:

$$y_t = 4y_{t-1} - 6y_{t-2} + 4y_{t-3} - y_{t-4}$$

Thus, we obtained, for each day, an estimated value. The correlation coefficient, calculated between statistical data and expected values is 0,85, which seems to be sufficient, taking into account the challenging problem of the exchange rate forecast.

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