3. ON SOLVING SOME TYPES OF MULTIPLE ATTRIBUTE DECISION-MAKING PROBLEMS

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Abstract

The paper considers various MADM problems with different attributes, including stochastic, fuzzy, numerical (cardinal) attributes. Some known MADM problems are presented and procedures to transform different problems into cardinal ones are proposed. For analysing stochastic and fuzzy MADM problems, informational measures as entropy and informational energy are used. Finally, a computer dedicated package is presented and an application is mentioned.

Keywords: MADM problems, SAW, TOPSIS, fuzzy criteria, entropy, informational energy, survey and pool data

JEL Classification: C13, C46

1. Introduction

A MADM (i.e. Multiple Attribute Decision-Making) problem can be formulated as follows (Andrasiu et. al., 1986; Chie-Bein Chen and Cerry M. Klein., 1997; Huang and Yoon, 1981; Swenson and McCahon, 1991; Vaduva and Resteanu, 2007; Yu, 1985; Zeleny, 1976; Zeleny,. 1982): there are n decision alternatives to be taken and there are m criteria or attributes used to determine the best (optimun) alternative decision. The attributes could be various. They could be real numbers, logical (extended) values (e.g. true false possible, less possible, etc), ranks (e.g. subjective marks assigned by human decidents) or linguistics appreciations or qualifying properties (such as good, bad, remarcable, etc.) or other ranking informations. In order to take a decision, to each criterion is associated a *"sense"* for selecting decisions, namely, the best decision is selected if its attribute has a minimum or a maximum value. The problem is to select the "best" decision alternative with respect to all criteria combined with *sense* requirements. A vector P represents the *importance* given by decidents to decision criteria; originaly we call them *relative weights*, but they could be transformed into probabilities.

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The data of a MADM problem can be represented as in the following table (Andrasiu, et. al., 1986; Chie-Bein Chen and Cerry M. Klein., 1997; Resteanu, Andreica and Vaduva, 2006; Swenson and McCahon, 1991):

Decision data

	C_1	C_2	...	C_m
A_1	a_{11}	a_{12}	...	a_{1m}
A_2	a_{21}	a_{22}	...	a_{2m}
.	.	...	.	...
A_n	a_{n1}	a_{n2}	...	a_{nm}
P	p_1	p_2	...	p_m
$sense$	$sense_1$	$sense_2$	...	$sense_m$

The entries $a_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$ define the $n \times m$ *decision matrix* **A=(a$_{ij}$)** The vector $P = (p_1, p_2, ..., p_m)'$ is usually a probability vector $(p_i > 0, \sum_{i=1}^{m} p_i = 1)$ specifying the "importance" of each criterion and the vector $sense = (sense_1, sense_2, ..., sense_m)'$ (see the comments from above), specifies the requirements for selecting the best decision alternative (i.e. if the best intended decision alternative requires a maximum value or a minimum value of the alternative).

The entries a_{ij} could be real numbers, linguistic qualificatives (Resteanu, Andreica. and Vaduva, 2006), logical values or any other elements from a specified ordered set.

There are various methods for solving a MADM problem, i.e. for determining an ordering or ranking of alternatives (i.e. $A_{(1)} \prec A_{(2)} \prec ... \prec A_{(n)}$) and then selecting the best decision alternative $A_{(n)}$.

The nature of a method is given by the entries a_{ij} (Andrasiu et. al., 1986; Chie-Bein Chen and Cerry M. Klein., 1997; Soung Hie Kim and Chung Hee Ham, 1999; Vaduva and Resteanu, 2007). If a_{ij} are deterministic real numbers then the problem is cardinal. The aim of the solution is to define the best order of alternatives, independent from the particular nature of the attributes. Sometime, the decision matrix can have a complex structure in the sense that its entries can have more indexes (Andrasiu. et. al., 1986; Huang and Yoon, 1981) as $a_{ijk}, 1 \leq k \leq d$. (The index k may refer to several human decidents involved in decision process). Such a problem is a MADM problem with several decidents. Practical MADM problems have different types of decision criteria (i.e. cardinal linguistic, stochastic, fuzzy, etc.). Sometime, before solving the MADM problem, all criteria must be transformed into cardinal ones. One aim of this paper is to illustrate how we can perform the transform of stochastic and fuzzy MADM criteria into cardinal criteria.

Any MADM method has to solve the following problems:

1. To make the set of criteria homogenuous; because attributes are of different types (real numbers, ranks, linguistics appreciations), i.e. they have different physical measures or meanings, we must transform them first into abstract real numbers. Before this we must scale the entries of A which are not numbers. Because the non-numbers entries of A belong to ordered sets, we will assign as a first step, positive integers to the values of attributes according to the relative order in the set. A special remark for fuzzy attributes (which are represented by fuzzy numbers). These could be transformed first into cardinal attrributes by the defuzzyfication operation. The same remark concerning stochastic entries of attributes, wich is discussed also later. Therefore, after the scaling, the matrix A contains only real numbers and it is a cardinal decision matrix.Then, the operation of normalization comes, which will allow us to perform correct mathematical operations to derive the MADM solution.
2. Since the weights represent relative measures of importance, it is necessary to determine the best probability (weight) vector P which assignes importance of decision alternatives. This step gives the best vector of importance of alternatives.
3. The third problem is to apply adequate mathematical models to perform the best ranking of alternatives. Therefore this step gives the final solution to the MADM problem.

In this paper we present some known methods for normalization and some methods for selecting the weights of importance and some methods for ranking. Then, we analyze the MADM problem with stochastic and fuzzy attributes.

2. On the normalization problem

There are various methods for normalization (Andrasiu et. al., 1986; Yoon and Wang, 1985; Yu, 1985; Zeleny, 1976, 1982). All of them transform the cardinal decision matrix A into a normalized matrix $R=(r_{ij}), 1 \le r_{ij} \le 1$. Here we summarize the vectorial normalization which is based on the general formula of a norm

$$r_{ij} = \frac{a_{ij}}{|a|_p}, a=(a_1,...,a_m), a_j^{(1,n)}, a_{ij} > 0, 1 \le i \le n, 1 \le j \le m \tag{2.1}$$

where: $|x|_p = (\sum_i X_i^p)^{1/p}$, with $p=1,2$. If $p=\infty$ then $|x|_p = max_i x_i$ and this normalization is called "by liniar transforms" (Andrasiu, M. et. al., 1986; Huang, C.L. and Yoon, K., 1981). Alternative formulae for normalizations by linear transforms are

$$r_{ij} = \frac{a_j^{max} - a_{ij}}{a_j^{max} - a_j^{min}}, if\ sense = max, \tag{2.2}$$

$$r_{ij} = \frac{a_{ij} - a_j^{min}}{a_j^{max} - a_j^{min}}, if\ sense = min, a_j^{min} = \min_i a_{ij}. \tag{2.3}$$

There are also other normalization methods but they are not frequently used.

3. Methods for evaluating the importance weights

3.1. The eigenvector method

The importance weights could be fixed up by the decidents. But in most cases the human decident defines only the relative importance of criteria specifying a matrix in the form

$$B = \begin{pmatrix} p_1/p_1 & p_1/p_2 & \cdots & p_1/p_m \\ p_2/p_1 & p_2/p_2 & \cdots & p_2/p_m \\ . & . & \cdots & . \\ p_m/p_1 & p_m/p_2 & \cdots & p_m/p_m \end{pmatrix} \qquad (3.1)$$

where: $b_{ij} = p_i/p_j$ is the *relative importance* of criterion i with respect the criterion j. (For instance b_{ij} means the importance given by the decident to criterion i with respect to criterion *j*). One should note that b_{ij} are related to the objective importance of criteria given by $P' = (p_1, p_2,..., p_m), p_i - unknown$ (P is a column vector!) and they satisfy properties

$$b_{ij} = \frac{1}{b_{ji}}, b_{ij} = \frac{b_{ik}}{b_{jk}}, 1 \le i, j, k \le m. \qquad (3.2)$$

From the preceeding formulae we find

$$BP = mP, \ i.e. \ (B - mE)P = 0, \qquad (3.3)$$

where: E is the unit matrix. Therefore, the best choice of P is an *eigen vector* of the matrix B. Therefore, the choice of P is obtained as follows (Andrasiu et. al., 1986; Huang. and Yoon, 1981):

- solves the characteristic equation $det(B - \lambda E) = 0$, giving λ_{max} the maximum *eigenvalue*;
- One solves the matrix equation $BP = \lambda_{max} P$, giving the corresponding eigen vector P which is the best importance weights vector.

One should note that b_{ij} are subjective numbers given by decidents. Some papers give hints on how to express the linguistics appreciations of the relative importance (such as: faible, reasonable small, small, big, reasonable big, etc) in terms of numbers (marks) from 1 to 9.

3.2. Least squares method

This method assumes that the sum of squares of errors between *theoretical* weights $b_{ij} p_j$ and empirical ones p_i is minimum, i.e.

$$\sum_{j=1}^{m}(b_{ij}p_j - p_i)^2 = min,\ p_i > 0, \sum_{i=1}^{m} p_i = 1. \qquad (3.4)$$

Being a minimization problem with constraints we use the Lagrange multipliers (Andrasiu et. al., 1986; Huang and Yoon, 1981) method, namely

$$\min_P \{\sum_{i=1}^{m}\sum_{j=1}^{m}(b_{ij}p_j - p_i)^2 + \lambda \sum_{i=1}^{m}(p_i - 1)\} \qquad (3.5)$$

which gives the following system of $m+1$ linear equations

$$\sum_{i=1}^{m}(b_{ij}p_i - p_i)b_{ii} - \sum_{j=1}^{m}(b_{ij}p_j - p_i) + \lambda = 0, 1 \le i \le m, \sum_{i=1}^{m} p_i - 1 = 0. \qquad (3.6)$$

From the system (3.6) we obtain $p_i = p_i(\lambda)$, then from the last equation we obtain the solution λ^* giving finally the best weights $p_i^* = p_i(\lambda^*), 1 \le i \le m.$

3.3. The Entropy Method

The previous methods used the subjective information given by decidents in matrix B. This method used the elements $a_{ij} > 0$ of the matrix A. The first step is to normalize (Andrasiu et. al., 1986; Huang and Yoon, 1981) the elements of the criteria (i.e. the columns of A) as

$$p_{ij} = \frac{a_{ij}}{\sum_{i=1}^{n} a_{ij}}, 1 \le j \le m. \qquad (3.7)$$

Then, we calculate the *normalized* entropy of the criterion j as

$$H_j = -k \sum_{i=1}^{n} p_{ij} \log p_{ij} \qquad (3.8)$$

where: $k = \frac{1}{\log n}$ is a normalizying constant which gives $0 \le H_j \le 1$ (resulting from Shannon's entropy). The measure of *diversity* of the values of the criterion j is then

$$d_j = 1 - H_j, 1 \le j \le m.$$

The importance weights given by this method are

$$p_j = \frac{d_j}{\sum_{i=1}^{m} d_j}, 1 \le j \le m, \sum_{i=1}^{m} p_j = 1. \qquad (3.9)$$

If, apart from the matrix A, the decident gives a priori importance weights $\lambda_1, \lambda_2, ..., \lambda_m, \sum_{i=1}^{m} \lambda_j = 1,$ then the method of entropy is improved giving the best importance weights as

$$p_j^0 = \frac{\lambda_j p_j}{\sum_{k=1}^{m} \lambda_k p_k}, 1 \le j \le m, \qquad (3.10)$$

satisfying the condition $\sum_{j=1}^{m} p_j^0 = 1$. Let us note that instead of entropy given by (3.8) we can use *Onicescu's informational energy* (see Onicescu, 1966; Onicescu and Stefanescu, 1979) given by

$$e_j = \sum_{i=1}^{n} p_{ij}^2, 1 \leq j \leq m \tag{3.11}$$

where: $\frac{1}{n} \leq e_j \leq 1$, (p_{ij} given by (3.7)). In this case we take

$$d_j = 1 - e_j, p_j = \frac{d_j}{\sum_{k=1}^{m} d_k}, \tag{3.12}$$

or if apriory subjective weights $\lambda_j, 1 \leq j \leq m$ are given, then,

$$p_j^0 = \frac{\lambda_j p_j}{\sum_{i=1}^{m} \lambda_i p_i}. \tag{3.13}$$

Note that the interpretation of informational energy is inverse to the interpretation of entropy, namely, when energy is maximum, the entropy is minimum and vice versa. This does not change the meaning of diversity.

4. Some methods for solving MADM problems

There is a large number of methods to solve MADM problems which depend of the information we have about the problem, such as:

- Methods for problems without any information about criteria or about decision alternatives.To this group belong the method of dominance, the method MAXIMIN and the method MAXIMAX.
- Methods using information about criteria (they are methods involving weights of importance of criteria). The set of this kind of methods is very large; we mention only some of them: conjunctive and disjunctive methods; lexicographic method and permutation method; the simple additive weighting method (SAW); the hierarchy additive weighting method; method of diameters; method of Onicescu; ELECTRE method; TOPSIS method (see later); method of hierarchic combinations; etc.
- Methods using information about decision alternatives. Examples of this type are: method LINMAP; iterative weighting additive method; method of multidimensional scalling.

In the papers (Resteanu, Andreica and Vaduva, 2006; Vaduva and Resteanu, 2008) we present a package and a site (called OPTCHOICE) devoted to learning and application of MADM methods. The methods implemented are the most known and used MADM methods. In the following we present some of these methods, illustrating two applications, and refer also to solving problems containing stochastic and fuzzy criteria.

In the next section, we present some known performant methods for solving cardinal MADM problems (Andrasiu et. al., 1986; Huang and Yoon, 1981; Yu, 1985; Zeleny, 1982).

4.1. Some cardinal MADM methods

Two of the best accepted MADM methods (Swenson, P.A. and McCahon, C.S., 1991) are SAW (Simple Additive Weighting) and TOPSIS (Technique for Order Preference by Similarity to Ideal Solution). We will present first these methods.

4.1.1. Simple Additive Weighting

Let us consider A=||aij|| the decision matrix, $a_{ij} \in R, a_{ij} \neq 0$. (If some $a_{ij} = 0$, then by a translation $a_{ij} := a_{ij} + h, h \neq 0$, we obtain $a_{ij} \neq 0$). Assume that all criteria have the *sense=max*; If a cardinal (numerical) criterion C_j has the *sense=min* then we change the entries from the column j of matrix a such as a_{ij} becomes $\frac{1}{a_{ij}}$ and the sense of criterion C_j is now *max.* If some attribute takes initially discrete *linguistic* values, they will be transformed conventionally by the decident into some real numbers (e.g. marks). In the following we assume that $a_{ij} \neq 0$. The SAW method assumes that the probability vector P=($p_1, p_2, \ldots, p_m$) of weights of importance is known.

The SAW method consists in the following steps (Andrasiu et. al., 1986; Huang and Yoon, 1981; Swenson and McCahon, 1991):

Step 1. Normalize elements of the decision matrix a, using one of the mentioned methods, obtaining the matrix $R = \| r_{ij} \|$.

As concerns the normalization, note again that $0 < r_{ij} \leq 1$.

Step 2. Calculate the values of the function $f : A \to R$ (A is the finite set of alternatives) as

$$f_i = f(A_i) = \frac{\sum_{j=1}^{m} p_j r_{ij}}{\sum_{i=1}^{m} p_j}, 1 \leq i \leq n \tag{4.1}$$

where: p_j are positive weights representing the relative importance of criteria (if p_j are probabilities then $\sum_{j=1}^{m} p_j = 1$).

Step 3. Order the values f_i obtaining the ordered sequence $f_{(1)} < f_{(2)} < \ldots < f_{(n)}$. To the sequence $f_{(i)}$ correspond ordering (or ranking) of the alternatives $A_{(i)}, 1 \leq i \leq n$. The best alternative is $A_{(n)}$ corresponding to $f_{(n)}$.

In general, the result of SAW does not depend on the normalization technique.

4.1.2. Technique of Order Preference by Similarity to Ideal Solution

This method consists in the following steps (Andrasiu et. al., 1986; Huang and Yoon, 1981; Swenson and McCahon, 1991):

Step 1. Normalize the decision matrix a, obtaining the normalized matrix R (as in section 3).

Step 2. Build-up the *weighted normalized matrix* $V = \| v_{ij} \|$ where:

$$v_{ij} = p_j r_{ij}, 1 \le i \le n, 1 \le j \le m.$$

Step 3. Build-up the *ideal positive solution* V^+ and the *ideal negative solution* V^- defined as

$$V^+ = (v_1^+, v_2+, ..., v_m^+), V^- = (v_1^-, v_2^-, ..., v_m^-)$$

where:

$$v_j^+ = \begin{cases} \max\limits_i v_{ij} & \text{if the criterion } j \text{ is a maximum one} \\ \min\limits_i v_{ij} & \text{if the criterion } j \text{ is a minimum one} \end{cases} \quad (4.2)$$

$$v_j^- = \begin{cases} \min\limits_i v_{ij} & \text{if the criterion } j \text{ is a maximum one} \\ \max\limits_i v_{ij} & \text{if the criterion } j \text{ is a minimum one} \end{cases} \quad (4.3)$$

Step 4. Calculate *distances* between weighted normalized entries v_{ij} and each of the ideal solutions (one uses the Euclideean distance), namely

$$D^+ = \sqrt{\sum_{j=1}^{m} (v_{ij} - v_j^+)^2}, \quad D^- = \sqrt{\sum_{i=1}^{m} (v_{ij} - v_j^-)^2}. \quad (4.4)$$

Step 5. Calculate for each alternative the *relative closeness to the ideal solution* as

$$Q_i = \frac{D_i^+}{D_i^+ + D_i^-}. \quad (4.5)$$

One should note that $0 < Q_i < 1.$

Step 6. Order the values of Q_i obtaining $Q_{(1)} \le Q_{(2)} \le ... \le Q_{(n)}$. The best alternative is $A_{(n)}$ corresponding to $Q_{(n)}$.

4.2. Combining several solutions

If we apply two MADM methods (i.e. based on SAW or on TOPSIS), we may obtain different solutions (maybe even different orderings of alternatives!). Practical experience shows that the *best* alternative $A_{(n)}$ is almost the same for any MADM method used. Finally we are interested to obtain one solution by combining the two ones. The following proposed procedure gives an unique solution.

Step 1. Formulate a new MADM problem with two criteria corresponding to the two solutions. The MADM decision matrix is an $n \times 2$ with elements $\alpha_{ij}, 1 \le i \le n, 1 \le j \le 2$ defined as

$$\alpha_{ij} = \begin{cases} f_i & if \quad j = 1 \\ Q_i & if \quad j = 2 \end{cases} \tag{4.6}$$

where: $f_i, Q_i, 1 \le i \le n$ are the values calculated by SAW and TOPSIS methods, respectively. The weights assigned to criteria could be 0.5 each, or could be specified by the decident.

Next, apply one of the cardinal methods presented in the previous section; here we propose the SAW method.

Step 2. Perform a normalization of the matrix $alpha = ¦¦ \alpha_{ij} ¦¦$, according to one of the procedures specified in the section 3, obtaining the matrix $R = || r_{ij} ||, 1 \le i \le n, 1 \le j \le 2$.

Step 3. Calculate the corresponding $F_i = F(A_i)$ by the formula (4.1);

Step 4. Order the values F_i obtaining $F_{(1)} < F_{(2)} < ... < F_{(n)}$.

To the $F_{(i)}$ corresponds the alternative $A_{(i)}$. Therefore, the ranking of alternatives is $A_{(1)} \prec A_{(2)} \prec ... \prec A_{(n)}$. The best solution is the alternative $A_{(n)}$.

Remark. Most of the cardinal MADM methods are based on calculation of a numerical function (such as f given by (4.1) or Q given by (4.5)) and the solution derives from ranking values that function.

Note. If for an initial MADM problem there are m solutions deriving from m different methods, given by values of some functions $g_j, 1 \le j \le m$ (of the type f or Q from above), then, a new MADM cardinal problem can be formulated as in Step 1 of this section, with the decision matrix $D = || d_{ij} ||, 1 \le i \le n, 1 \le j \le m$ defined as

$$d_{ij} = g_j(A_i) \tag{4.7}$$

with the probability vector P either fixed-up by the decident, or determined as in section 3 by using the matrix B of preferences , also given by the decident. Then, an algorithm (such as SAW or TOPSIS) can be applied obtaining the final *combined* solution.

5. Processing of stochastic entries

Assume that an entry a_{ij} is a real random variable X specified either as a discrete probability distribution in the form

$$X : \begin{pmatrix} a_1, & a_2, & ... & a_k \\ p_1, & p_2, & ... & p_k \end{pmatrix}, a_i \in R, p_i > 0, \sum_{i=1}^{k} p_i = 1,$$

or as a continuous probability distribution given by its probability density function (p.d.f.) $f(x)$ in the form

$$X \sim> f(x), x \in [a,b], -\infty < a < b < \infty, \int_a^b f(x)dx = 1.$$

In other words, the elements a_i, p_i in the discrete case or $a, b, f(x)$ in the continuous case are given, for a stochastic MADM problem. Many of the methods for solving MADM problems are reduced to solving cardinal (deterministic) problems. This idea is applied here for solving stochastic MADM problems in the sense that we first transform a stochastic criterion into a cardinal one. More precisely, the stochastic decision entry a_{ij} of the matrix $\| a_{ij} \|$ is transformed into a cardinal (deterministic) entry h_{ij}.

The procedure consists in the following steps:

Step 1. A stochastic entry X is transformed into a *standardized* stochastic entry Y in the form

$$Y = \frac{X}{\sigma} \tag{5.1}$$

where: $\sigma^2 = Var(X)$ is the variance of X (i.e. σ is the *standard deviation* of X).

In the discrete case we have

$$m = E[X] = \sum_{i=1}^{k} a_i p_i, \sigma^2 = E[(X-m)^2] = \sum_{i=1}^{k} p_i a_i^2 - m^2$$

and in the continuous case we have

$$m = E[X] = \int_a^b x f(x)dx, \sigma^2 = E[(X-m)^2] = \int_a^b x^2 f(x)dx - m^2,$$

assuming that m and σ^2 exist.

According to (5.1) the discrete distribution of Y is

$$Y : \begin{pmatrix} b_1, & b_2, & ..., & b_k \\ p_1, & p_2, & ..., & p_k \end{pmatrix}, b_i = \frac{a_i}{\sigma} \tag{5.2}$$

and in the continuous case the probability density function of Y is

$$Y \sim> g(y) = \sigma f(\sigma y),\ y \in [\alpha, \beta], \alpha = \frac{a}{\sigma}, \beta = \frac{b}{\sigma}. \tag{5.3}$$

The *range* of standardized r.v. Y is $r_a = \beta - \alpha$, where, in the discrete case $\alpha = b_1, \beta = b_k$. If the range is ∞ then one finds a finite interval $[\alpha, \beta]$ such as: $P(Y \in [\alpha, \beta]) = p,$ with p close to 1 (for instance, if $Y \mapsto Normal(m, \sigma)$ then $\alpha = m - 3\sigma, \beta = m + 3\sigma$ with $p = 1 - 10^{-4} \approx 1$. For distributions like $Exp(\lambda)$ or $Lomax(\theta, a)$ the range is $\alpha = 0,$ and β is defined using the cdf $F(x)$ such as $F(\beta) = 1 - p, p \le 10^{-4}$).

Step 2. The next proposed step is to assign to a stochastic entry Y in the decision matrix the *information* contained in the corresponding probability distribution. This information can be represented either by *Shannon's entropy* or by *Onicescu's informational energy* of Y.

In the discrete case the entropy of Y is

$$h = -\sum_{i=1}^{k} p_i \log p_i \quad (5.4)$$

and the informational energy is

$$e = \sum_{i=1}^{k} p_i^2. \quad (5.5)$$

In the continuous case the entropy of Y is

$$h = -\int_{\alpha}^{\beta} g(y) \log g(y) dy = -\sigma \int_{a\sigma}^{b\sigma} f(\sigma x) \log(\sigma f(\sigma x)) dx \quad (5.6)$$

and the informational energy is

$$e = \int_{\alpha}^{\beta} g^2(y) dy = \sigma^2 \int_{a\sigma}^{b\sigma} f^2(\sigma x) dx. \quad (5.7)$$

Now, the decision matrix of the problem has entries corresponding to the criterion C_j in the form h_{ij} or e_{ij}, i.e. it is a cardinal one.

Note. In the continuous case, the formula (5.7) says that the informational energy of a random variable X having the density $f(x)$ is

$$e = E[f(X)] \quad (5.8)$$

and the entropy is

$$h = -E[\log f(X)]. \quad (5.9)$$

In Vaduva (1989) the following is presented:

Proposition. 1 The probability density function $g(v)$ of the random variable $Y = f(X)$, is

$$g(v) = -vA'(v), \quad A(v) = mes\{x \mid f(x) \geq v\}. \quad (5.10)$$

For some particular distributions the p.d.f of $Y = f(X)$ are (Vaduva, I., 1989):

1.1. *For exponential distribution* $Exp(\lambda)$ *we have* $g(v) = \frac{v}{\lambda}$, i.e. $V = f(X)$ is uniformly distributed on $[0, \lambda]$. This suggests an *inverse* method for simulating $X \sim> Exp(\lambda)$, namely:

- generate V uniform on $[0, \lambda]$;
- calculate $X = -\frac{1}{\lambda} \log(\frac{V}{\lambda})$.

1.2. For the normal $N(0,1)$ distribution we have $g(v) = 2[-2 \log(\sqrt{2\pi} v)]^{-1/2}$ for $0 \leq v \leq \frac{1}{\sqrt{2\pi}}$.

1.3. For the bivariate normal $N(0, \mathbf{I})$ (**I**=the 2×2 unit matrix),the p.d.f. of $V = f(X_1, X_2)$ is $g(v) = 2\pi$, i.e. V is uniform.

Because the proof of this fact is related to the well known Box-Muller method for simulating this distribution, one obtains the following method of simulating this distribution, namely: if V in uniformly distributed on $[0,\frac{1}{\sqrt{2\pi}}]$ then, the random point $(X_1,X_2)'$ on the curve $f(X_1,X_2)=V$ is normally $N(0,I)$ distributed.

In Vaduva, I. (1989) the pdf $g(v)$ is calculated for a bivariate $t-$distribution and for a bivariate Cauchy distribution; note that $g(v)$ is not allwys a uniform density.

Densities $g(v)$ could be used to calculate entropy and informational energy.

By direct calculation (Onicescu, 1966; Onicescu and Stefanescu, 1979), one can obtain informational energy for particular distributions, namely:

- for the normal $N(m,\sigma)$ distribution we have $e=\frac{1}{2\sqrt{\pi}\sigma}$;
- for the exponential $Exp(\lambda)$ distribution we have $e=\frac{\lambda}{2}$;
- for standard Cauchy distribution we have $e=\frac{1}{2\pi}$.

If the calculation of e or h is easier with (5.8) and (5.9) then the p.d.f. of $f(X)$ should be used, as an alternative to (5.6) and(5.7).

The stochastic criteria in a MADM problem could be processed in two ways:

1^0. Consider in the decision matrix for a stochastic criterion C_j the *information per range* defined as

$$a_{ij}=\frac{h_{ij}}{\beta-\alpha} or\ a_{ij}=\frac{e_{ij}}{\beta-\alpha}. \tag{5.11}$$

Now instead of column j in the initial decision matrix, include two columns: one with the mean value of the random criterion C_j conserving the initial sense $sense_j$, and another column with entries from (5.7) with $sense=max$ for entropy or $sense=min$ for energy. Therefore, if the initial decision matrix (having m criteria) contains s stochastic criteria and $m-s$ cardinal criteria, then the decision matrix will contain $m+s$ cardinal criteria. This cardinal problem will give the MADM solution of the initial problem.

2^0. An alternative procedure is to solve first separately the cardinal problem of $2s$ criteria deriving from the initial s stochastic criteria, as mentioned in the previous pct.1^0. Then, build up a new MADM problem with the initial non-stochastic $m-s$ criteria and fill it in with a new criterion consisting of the solution of the cardinal MADM problem of $2s$ criteria deriving from the s stochastic criteria. The solution of the initial MADM problem will be the solution of the last cardinal problem containing $m-s+1$ criteria.The cardinal attributes of the last criterion are the *ranks* of alternatives resulted for processing stochastic criteria, or the *function* g (of the form f or Q) which orders the set of stochastic criteria.

6. On processing fuzzy criteria

6.1. Introduction

A fuzzy set A, (Vaduva. and Albeanu, 2004; Zimmermann, 1994) is defined in terms of a referential X (a *crisp set*) by its *membership function* $\varphi_A(x) \in [0,1], \forall x \in X$, which gives the *degree of ownership* of $x \in X$ in A. When A is a discrete set, the degree of ownership is specified to each element of A. Entries of fuzzy criteria of a MADM problem could be expressed in two ways: (1) as linguistic variables, (2) as fuzzy numbers (with the referential= $X = R$).

(1). Some details on *linguistics variables.* Such a variable is defined as the quintuple $(x, T(x), X, G, M)$ where x is the name (label) of the variable, $T(x)$ is the *set of terms* of x, X is the *univers* set, G is a *grammer* specifying the syntactic rules to produce a *linguistic value* and M is a *semantic* rule to give the meaning of a linguistic value. For instance the meaning of a linguistic value t, denoted by $M(t)$, is a *membership value* associated to t.

Example. Let $X = \{1..100\}$ and $v =$ the linguistic value labeled as *age*, and the set of linguistics values $T(v) = \{young, adult, old\}$. (These values are defined in terms of subintervals of X.) A term can be modified by a *modifier* that modifies its meaning; i.e. very old, very young, etc. Modifiers are expressed in terms of *arithmetic operations* such as *power*. For instance, if a is a *primary term* ($a \in [0,1]$, i.e a is the membership value of term) then

$$very = \sqrt{a}; very\ very = \sqrt{\sqrt{a}}; less = a^2. \qquad (6.1)$$

Therefore, the operation of power, $a^p, p \in (0,\infty)$ defines a large family of modifiers; note that for $p = \infty$ the modifier can be named *exact* since all membership degrees less than 1.0 are *suppressed.*

(2). Some details regarding the modifier of a fuzzy number $\varphi_A(x)$, which refers to an attribute entry of a MADM matrix, as a fuzzy set A which is a continuous *membership function* are given in (Vaduva, I. and Albeanu, Gr., 2004; Zimmermann, H.J., 1994).

For instance, the power modifier is in the form

$$m_p(A) = \int_X \varphi_A^p(u)du, \qquad (6.2)$$

and the *lag* modifier is

$$m_l(A) = \int_a \varphi_A(x - s)dx, \qquad (6.3)$$

where: s is the seize of lag; The fuzzy numbers used are of different types but the most familiar are:

triangular fuzzy number denoted by $\Delta(a,m,c)$ as defined by the membership function

$$\mu_{\Delta}(x)=\begin{cases}\frac{1}{m-a}x-\frac{a}{m-a}, x\in[a,m]\\ \frac{1}{m-c}x-\frac{c}{m-c}, x\in[m,c]\\ o, otherwise\end{cases} \quad (6.4)$$

trapezoidal fuzzy number denoted by $t(a,b,c,d)$, as defined by the membership function

$$\tau_{t(a,b,c,d)}(x)=\begin{cases}\frac{x-a}{b-a}, a\leq x\leq b,\\ 1, b\leq x\leq c,\\ \frac{x-d}{c-d}, c\leq x\leq d\\ o, otherwise\end{cases}. \quad (6.5)$$

For processing fuzzy attributes we propose the use of the *precision index* based on the *Hartley entropy* given by (6.7) bellow. It is defined in terms of the *cut of level* α *of a fuzzy set* A denoted by $[\varphi]_{\alpha}$ which is the crisp set

$$[\varphi]_{\alpha}=\{x\mid\varphi(x)\geq\alpha\}, \quad (6.6)$$

where: $\varphi(x)$ is the membership function of A. The (Hartley) *entropy of the fuzzy set* A (as defined by Higashi and Klir in 1983 cited in Vaduva and Albeanu, 2004) is

$$H(A)=\int_0^1 log_2(|[\varphi]_{\alpha}|)d\alpha, \quad (6.7)$$

where: $|A|$ is the cardinal of the crisp set A. This *fuzzy entropy* is used to define a *precision index* of the fuzzy set A represented by its membership function which is given as

$$pr_H(A)=pr(A)=e^{-H(A)}, \quad (6.8)$$

having the following properties

$$pr(A)\in[0,1]; pr(A)=1\leftrightarrow card(A)=1; if\ A\subseteq B\ then\ pr(A)\geq pr(B). \quad (6.9)$$

Similarly to stochastic criteria we can introduce also the *fuzzy energy* defined as

$$E(A)=\int_0^1(|[\varphi]_{\alpha}|)^2 d\alpha \quad (6.10)$$

with the corresponding precision index

$$pr_e(A)=e^{-E(A)}. \quad (6.11)$$

One should note that similarly to $pr_H(a)$ the $pr_e(A)$ satisfies the conditions

$$pr_e(A)\in[e^{-1},1], pr_e(A)=e^{-1}\leftrightarrow card(A)=1; if\ A\subseteq B\ then\ pr_e(A)\leq pr_e(B).$$

6.2. Ordering of alternatives with respect to fuzzy criteria

There are several known methods for ranking alternatives by fuzzy criteria. We present first two of them.

The MAXIMIN method. This method is based on the order relation for fuzzy numbers defined as:

Definition 1 For the fuzzy numbers F,G the relation $\geq$ is defined as

$$\varphi_{\geq}(F,G) = \sup_{u} min\{\varphi_F(u), \sup_{v \leq u} \varphi_G(v)\} = \sup_{u,v,u \leq v} min\{\varphi_f(u), \varphi_g(v)\}. \quad (6.12)$$

and the relation $<$ is defined as

$$\varphi_{<}(F,G) = 1 - \varphi_{\geq}(F,G). \quad (6.13)$$

The MAXINNMIN method. If the decision matrix is in the form $A = (\mu_{ij})$ where: μ_{ij} are fuzzy numbers, then the MAXIMIN method selects the alternative A_{i_0} such as

$$\mu_{i_0 j} = \max_i \min_j \mu_{ij}. \quad (6.14)$$

The SAW fuzzy method. This method is based on the algorithm described in the subsection 4.1.1. The "cardinal" operations in that subsection are made now in terms of operations with fuzzy numbers defined as

$$\varphi_{\mu+\eta}(x) = \mu(x) + \eta(x) - \mu(x)\eta(x); \varphi_{k\mu}(x) = k\varphi(x), \forall k \in [0,1]. \quad (6.15)$$

The ranking of alternatives is done according to order relations defined by (6.12),(6.13).

The TOPSIS fuzzy method. In order to apply the TOPSIS method to criteria expressed as fuzzy numbers it is necessary to use also the *distance* between fuzzy numbers. We remind first the *Hausdorf distance on* R^n.

Definition 2 Let A, B be sets in R^n. If $\| a \|$ is the Euclidian norm of $a \in R^n$, then the following distances are Housdorf distances:

$$d(a,B) = \inf_{b \in B} \| a - b \| \quad (6.16)$$

$$d(b,A) = \inf_{a \in A} \| a - b \| \quad (6.17)$$

$$d(A,B) = max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\} \quad (6.18)$$

Definition 3 Distances between fuzzy numbers, coresponding to Housdorf distances are:

$$d_\infty(\mu,\eta) = \sup_{\alpha \in [0,1]} d([\mu]_\alpha, [\eta]_\alpha), \quad (6.19)$$

$$d_1(\mu,\eta) = \int_0^1 d([\mu]_\alpha, [\eta]_\alpha) d\alpha, \quad (6.20)$$

$$d_r(\mu,\eta) = \left[\int_0^1 d^r([\mu]_\alpha, [\eta]_\alpha d\alpha \right]^{\frac{1}{r}} \quad (6.21)$$

(Notations for fuzzy numbers as I (6.5) are used).

Using distance between fuzzy numbers, all operations involved in the algorithm TOPSIS are defined for **all** attributes represented as fuzzy numbers.

New ways to handle fuzzy criteria. As we mentioned before, there are MADM problems with s fuzzy criteria and $m - s$ cardinal entries. In this case, one way could be to use all algorithms for cardinal methods, using fuzzy operations and orderings when required. But could be also three possible alternatives:

(a). To perform a separate ordering of alternatives with respect to fuzzy criteria obtaining an order of alternatives described in terms of fuzzy numbers of attributes. Then, perform a *defuzzyfication* operation for obtaining a cardinal criterion. The $m-s$ cardinal criteria with the last criterion (obtained from fuzzy criteria) will form a cardinal MADM problem with $m-s+1$ criteria.

(b). To perform the operation of defuzzyfication of all fuzzy entries, obtaining a cardinal MADM problem with m criteria. The solution of the last problem is the solution of the original problem. In both situations we propose the following way to use fuzzy criteria, similar to the stochastic criteria, namely:

(c). *To attach to every fuzzy citerion an INFORMATIONAL criterion* defined in terms of precision index. Thus, to each fuzzy criterion will correspond two criteria: one the criterion itself and the other criterion a cardinal one, say, K defined in terms of precision index with the *sense=max* for $pr_H(K)$ and the *sense= min* for $pr_e(K)$. After defuzzyfication of fuzzy criteria one obtains a decision problem with cardinal entries and $m+s$ criteria.

There are different **defuzzyfication procedures.** We specify only two which are suitable for fuzzy (continous or discrete) attributes in MADM problems:

cog = center of gravity defined as

$$cog(A) = \frac{\int_X \varphi_A(x)x dx}{\int_X \varphi_A(x)dx}, or\ cog(A) = \frac{\sum_{j=1}^{q} \varphi_A(y_j)y_j}{\sum_{j=1}^{q} \varphi_A(y_j)}, \tag{6.22}$$

coa = center of area defined as

$$\int_{\underset{X}{inf}}^{coa(A)} \varphi_A(x)dx = \int_{coa(A)}^{\underset{X}{sup}} \varphi(x)dx, \quad \sum_{j=1}^{coa(A)} \varphi_A(y_j) = \sum_{j=coa(A)}^{q} \varphi_A(yj). \tag{6.23}$$

Note that $coa(A)$ is a kind of empirical *median* in the discrete case, therefore it can be calculated using this similitude.

In practice it might happen that for some criteria some of their values be cardinal and the other ones fuzzy. Then, one way to handle such a criterion is to do first defuzzyfication of fuzzy attributes and so the whole criterion becomes cardinal.

7. On two applications of MADM

7.1. Introduction

A reasearch project *Excellency Level Tools for Multi-Attribute Decision Making Field's Promotion* developed by the National Institute for Research and Dervelopment in Informatics of Romania, the University of Bucharest and the Academy of Economic Studies in Bucharest, was developed during 2005-2008 with the following components: a Computing Package for analyzing and solving MADM problems; a module (e-course) for learning MADM applications; and a module for certifying (testing) the user of the package. All these functions are incorporated in an IT product called *OPTCHOICE* (see reference Resteanu, Andreica. and Vaduva., 2006; Vaduva and Resteanu, 2008). Types of MADM methods implemented in OPTCHOICE are: MAXIMIN,MAXIMAX, lexicographic, with ordinal preferences, conjuctive, disjunctive, elimination by aspects, permutation, linear assignment, SAW, Hierarchical Additive Weighting (HAW), diameters, Onicescu, ELECTRE, TOPSIS, TODIM, Pareto, Saphier-Rusu, minimization of deviation, scores, hierarchical combinations, LINMAP, multidimensional scaling with ideal solution, Saaty, etc. Using some algorithms from OPTCHOICE we performed two applications which are presented for short in the following.

7.2. Ordering faculties of the University

A University consists of 19 faculties and they must be ordered (ranked) according to 12 criteria. The faculties are identified by a *code:1,2,...,19* and the criteria with the corresponding *sense* and *weights vectors* are presented in the following table

List of criteria, sense and weights vectors

No	Criteria	Sense	Weights
1	Total number of state students	Max	0.09
2	Number of students paying fees per member of staff	Max	0.07
3	Total number of students per member staff	Max	0.12
4	Average salary per professor	Min	0.12
5	Average salary per reader (senior lecturer)	Min	0.06
6	Average salary per lecturer	Min	0.06
7	Percentage of professors from total staff	Min	0.10
8	Percentage of readers from total staff	Min	0.07
9	Percentage of lecturers from total staff	Min	0.07
10	Surplus/deficit of the faculty (mil. currency)	Max	0.12
11	Number of auxiliary staff (secretaries, technicians.)	Min	0.06
12	Average salary per member of auxiliary staff	Min	0.06

In the table "surplus" is a positive amount of money while "deficit" is a negative one.

The solution requird by the University senate is to find a decreasing order of the faculties. Note that according to weights assigned, criteria 1, 3, 4 and 10 are the most important, while salaries for low level staff and auxiliaries are less important. The figures in the table were collected from the files of faculties. The number of students is adjusted according to some coefficients of the Ministry of Education (e.g. one student from the faculties of natural sciences is 1.3 to 1.6 as compared to a student from humanistics who has the coefficient = 1).

The following table presents the decision data.

Decision data

No.	1	2	3	4	5	6	7	8	9	10	11	12
1	2675	1.34	45	9383	5341	4921	0.29	0.17	0.35	2.67	13	1335
2	1950	0.72	35	7907	2462	2139	0.29	0.15	0.34	0.98	11	1325
3	3324	0.67	31	9680	5680	3341	0.17	0.17	0.32	-0.27	14	1100
4	1857	0.55	28	8901	4710	2552	0.17	0.10	0.28	-0.21	8	1436
5	606	1.68	33	6534	2453	2537	0.27	0.13	0.36	0.13	2	1269
6	4143	0.15	6	6336	3546	2284	0.27	0.19	0.30	-11	76	1033
7	1007	0.41	21	5602	3477	1714	0.42	0.07	0.17	-0.85	3	2332
8	331	0.31	13	11467	3055	1654	0.12	0.13	0.5.	0.68	1	2008
9	3125	0.11	8	5367	3087	2049	0.26	0.22	0.30	-0.47	78	926
10	2323	0.56	31	7358	3843	2033	0.17	0.05	0.29	-2.78	31	926
11	1457	0.37	15	4912	2963	2028	0.41	0.21	0.21	-2.13	5	990
12	312	0.03	9	110	1754	1965	0.01	0.12	0.24	-0.48	1	714
13	2744	0.20	20	5738	2300	1910	0.24	0.21	0.31	-6.12	7	1138
14	2854	0.18	10	4757	3367	2419	0.37	0.15	0.19	-8.27	21	1049
15	3370	0.09	7	5255	2875	1874	0.29	0.17	0.26	-5.87	27	1056
16	662	0.75	24	5027	3850	2594	0.14	0.18	0.14	-0.65	4	1072
17	4049	0.73	13	5067	2806	1486	0.19	0.21	0.34	-5.84	14	1044
18	1314	0.11	9	4508	2526	1614	0.23	0.22	0.34	-2.42	72	1165
19	1694	0.95	21	5702	2688	1443	0.13	0.17	0.27	1.79	6	1536

Since all input data are *cardinal* there were applied only two of the methods reported (Swenson and McCahon, 1991) as beeing good with respect to their complexity, namely SAW and TOPSIS. The results obtained by these methods are presented in the following table. The order of the faculties resulting from the SAW metdod is serial, while the order resulted from TOPSIS is somehow different. Although, **the first two faculties** have the same rank as well as faculties **9-19.** Overall we conclude that SAW and TOPSIS methods give a similar order of the alternatives. One can see that criterion 10 (surplus/deficit) having a high weight ($p_{10} = 0.12$) is very important for the ordering of alternatives. Furthermore, alternatives 16-19 have close values for functions f and Q. Note also that alternatives 1 and 2 have the same rank for both methods.The best and the last ranked alternatives have the same rank for both methods, which is considered the main aim of MADM applications. By comparying values of functions f and Q, which have close values for alternative which have close, different ranks, one can accept the ranking given by the method for wich differences of figures of the corresponding function are comparably larger (in our case the SAW method).

Results of a MADM application

Faculty	f of SAW	Rank of f	Q of TOPSIS	Rank of Q
1	0.574702	1	0.706133	1
2	0.48130	2	0.403615	2
3	0.416766	3	0.365562	4
4	0.388023	4	0.358544	5
5	0.377665	5	0.356807	3
6	0.316790	6	0.339552	7
7	0.309472	7	0.347743	6
8	0.299454	8	0.329757	8
9	0.295305	9	0.325576	9
10	0.288292	10	0.317848	10
11	0.261160	11	0.311862	11
12	0.240736	12	0.305434	12
13	0.193280	13	0.274758	13
14	0.133284	14	0.257312	14
15	0.073655	15	0.233541	15
16	0.038269	16	0.206828	16
17	0.021641	17	0.204988	17
18	-0.084386	18	0.157250	18
19	-0.215411	19	0.155256	19

8. On applying the MADM method for processing survey and pool data

In this section we refer in short to an application of MADM to a problem requested by a privat TV Channel from Romania, Realitatea TV. There were eight fields of social and political activities. The problem was to select the most representative Romanian personality in one year, with the highest notoriety in each field. (So far there were four edditions of this project; we refer only to the one which was proposed by us, but unfortunately not used.) A group of selectors defined a list of 20 persons from each field. The three criteria were the following: the ranking given by specialists in each particular field; the rakning given by members of the association of each field; the ranking given by the whole population watching TV (i.e. people, TV viewers).

Suggestions for collection of data. The survey and pool data were collected referring to the starting list established by the groups of selectors for each field. The appreciations by the individuals of each "criterion" were recorded in terms of marks (integer numbers from 1 to 10 = the largest mark). In fact, there were eight MADM problems with three criteria each and collected data consisted in empirical probability distributions by marks for each of the alternative and for each field. (All attributes are therefore empirical distributions.) These distributions of each attribute are transformed into *fuzzy numbers* as follows: the minimum mark a with positive frequency and the maximum mark b of positive frequency define the interval (support) of the membership function; if the empirical distribution (e.d.) has only one mode, then that modal point is the peak of a triangular fuzzy number, while if the e.d. has several modes, then the two extreme modes (minimal and maximal) are peaks of trapezoidal numbers. As the data collected from the population are of a very large sample size, there were considered for each alternative only the *percentage* extracted from pool data, i.e. the percentage of respondents who prefered that alternative. (Note that the public could include also some other alternatives, but as they are expected to have very small frequencies, they will be ommited!) Thus, from the people's vote, we associate to each alternative having probability (frequency) $= p$ a *binomial distribution* $Bin(n, p)$, where n is the number of participants to the vote. As n is very large, according to the *Central Limit Theorem* the number of people X who gave the vote to that alternative is *normally distributed* $N(np, \sqrt{np(1-p)}), \sigma = \sqrt{np(1-p)}.$ The fuzzy number associated to that attribute is a symmetrical one with support $[np-3\sigma, np+3\sigma],$ and the peak is $np.$

Therefore, one of two ways may be chosen: to associate to attributes (i.e. entries) in the decision matrices fuzzy numbers as described, or to associate to attributes of the first two criteria probability distributions (selected according to a *test of goodness of fit*) and for the third criterion (the vote of people), the normal distribution as mentioned.

After transforming fuzzy entries or stochastic entries into cardinal ones, the processing might be performed with SAW and TOPSIS. Even if the procedure was not applied to real data, we expect that the best alternatives, for these two methods,will be similar to those in the case of section 7.1.

Finally we underline that even the technique described in this section was not applied, it might be interesting as a methodological approach, mainly to collecting stochastic or fuzzy data.

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