4. JOINT MODELLING OF S&P500 AND VIX INDICES WITH ROUGH FRACTIONAL ORNSTEIN-UHLENBECK VOLATILITY MODEL

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Abstract

In this paper, we study the joint modelling problem of S&P500 and VIX indices, under rough volatility dynamics by a stochastic model with continuous paths. Our aim is to improve the future values' forecast of S&P500 index using the VIX index estimates. The present study is built on the estimation with the rough volatility models of the noise component which is included in financial models. The main stylized facts of the volatility can be captured well by fractional Brownian motions with a Hurst index, lower than 0.5. The H parameter governs the realized volatility roughness of time series. In the rough volatility approach, the Hurst exponent H is estimated by using the scaling properties of the volatility series. We describe the log-volatility of S&P500 index using a rough fractional Ornstein-Uhlenbeck model. The VIX index is a measure of the market's expected volatility on the S&P 500 Index. When the rBergomi model is empirically calibrated to daily data of the proxy, realized volatility and the VIX index, it is found that the VIX index is rough with $H < 0.3$ and consistent with daily implied volatility. The findings suggest that the VIX index is consistent with daily implied volatility of S&P500 and also rescaled version of the VIX index can be used to model the volatility process of S&P500 index. Finally, price estimates of S&P500 can be properly approached by using a Rough Fractional Ornstein-Uhlenbeck model of VIX index which is an implied volatility process.

Keywords: rough volatility, fractional Ornstein-Uhlenbeck process, volatility estimation, rBergomi model, S&P500 price model

JEL Classification: G17, C58, E37, C22, F47

1. Introduction

There are many factors that affect the behavior of financial markets. When we want to model the behavior of a financial asset mathematically over time, we cannot usually gather enough detailed information or define the evaluation mechanism of the financial asset precisely. In this case, stochastic models may be offered as useful tools to satisfy this aim. Stochastic models can be used to predict future value of interested asset price. Generally, financial

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assets are modelled with stochastic differential equations driven by Brownian motion. However, the empirical studies show that financial time series have long range memory property. The long memory property can be controlled using fractional Brownian motion with $H > 0.5$ (Comte, Coutin and Renault, 2012). Another stylized fact of financial time series is that the series come back to their long range mean level over time. This type of behavior is known as mean-reversion and often modeled by the Ornstein-Uhlenbeck process (OU). Barndorff-Nielsen and Shephard (2001) used an OU process to model the stochastic volatility of financial assets. If the data are observed with high frequency, the semimartingale structure of series deteriorates. The reason for this phenomenon is the market microstructure noise. Heston or Bates stochastic volatility models cannot reproduce some important empirical facts of the observed volatility surface.

For the first time, Gatheral *et al*. (2018) propose a "Rough Fractional Stochastic Volatility" (RFSV) model, in which the process of log-volatility is modelled in terms of a fractional Ornstein-Uhlenbeck(fOU) process with $H < 0.5$. Gatheral's model is highly consistent with empirical estimates of volatile time series. In the RFSV model, the volatility process $\sigma_t = exp(X_t)$, where: X_t is the solution of the (fOU) process. Gatheral *et al*., (2014) and Bennedsen *et al*., (2016), using the high-frequency price data, obtain the value of $H < 0.5$. In the study of Gatheral *et al*. (2018), the realized variance estimates, taken from the Oxford-Man Institute of Quantitative Finance Realized Library 2 (http://realized.oxford-man.ox.ac.uk/data/download), are used as volatilities. Furthermore, they state that the estimation of the Hurst exponent, H, is robust across time, scales and markets. Volatility is a proxy for the magnitude of price changes and acts as a stochastic process. The VIX volatility index is designed to measure the market's expectation for 30 days volatility, implied by the S&P 500 index option price. The VIX shows the annualized square root of the price of a contract with payoff equal to $log(S_{t+\Delta}/S_t)$), where: $\Delta = 30$ days and S_t denotes the value of the SPX. Furthermore, S&P500 implied that the correlation index (CIX) can also be used for predicting the S&P500 index return. The CIX index can be used for reflecting market's overall systematic risk. Since the volatility itself is not observable, we need a proxy for the volatility process. Absolute and squared returns are often used for tvolatility proxy process. To get a better estimator for volatility, we can use EWMA, quadratic variation or realized kernel estimator. In general, the VIX and S&P500 indices have a negative correlation. To model the dynamics of the VIX index, different models are proposed in literature, for example, the log-normal Ornstein-Uhlenbeck (OU) diffusion by Mencia and Sentana (2013), the mean-reverting process by Kaeck and Alexander (2013), etc. Stochastic volatility models are often based on the Markovian assumption for the underlying asset price process. However, the rough volatility approach assumes that the instantaneous volatility is driven by fractional Brownian motion. Bennedsen, Lunde, and Pakkanen (2016) examined daily volatility measurements of individual US stocks and found lower values for H (between 0.05 and 0.2) and volatility roughness. The aim of this paper is to obtain a continuous time model of rough volatility process $(\sigma_t)_{t\geq 0}$ and then to simulate the S&P500 index values. This approach requires our model to involve both roughness (irregular behavior at short time scales) and persistence (strong dependence at longer time scales). The first contribution of our empirical study is to show that observed time series are rough, persistent, and non-Gaussian. The second contribution is that rescaled VIX index can be used for modeling the volatility of the S&P500 index.

The main contribution of this paper is to propose the joint modeling of dynamics of the VIX index and the underlying financial asset S&P500 index using rough volatility models (non-Markovian models). In this way, we aim to compute fair values of the SP500 index. We model the dynamics of the log-volatility process using a Rough Fractional Ornstein-Uhlenbeck Volatility process and we use the (EWMA) model to derive the proxy volatility process of the spot (squared) volatility of a day of the S&P500 index, because this method may capture properties as heteroskedasticity and volatility clustering. The roughness of the realized volatility is assessed by estimating the Hurst parameter, $H \in (0,1)$. We contribute to verify the rough behavior of volatility by showing that the statistical estimation of the Hurst index, H, is lower than 0.5 and also by showing that the volatility is more irregular than a standard Brownian motion.

The paper is organized as follows. In section 2, we review fundamental properties of the Ornstein-Uhlenbeck process and fractional Brownian motion. Section 3 includes some results on generation of volatility of financial assets and realized variance and rough volatility. Section 4 presents the model, applied to real S&P500 and VIX index data. Section 5 includes the conclusions.

2. The Fractional Ornstein-Uhlenbeck Process

2.1. The Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process (OU), introduced by Uhlenbeck and Ornstein (1930), is a stochastic process which reverts to its long-term mean over time. The process (OU) can be considered as continuous time process similar to the discrete-time AR(1) process and it satisfies the following stochastic differential equation,

$$dV_t = \kappa(\mu - V_t)dt + \sigma(t, V_t)dW_t$$

where: $V_0 = v_0$ is known, $(W_t)_{t\geq 0}$ is a standart Brownian motion, $\kappa(\mu - V_t)$ drift and $\sigma(t, V_t)$ are volatility coefficients, respectively. This equation is known as the Vasicek model, which is given by Vasicek (1977). $V_0 = v_0$ is known and $dW_t = \sqrt{dt}\, Z_t$ with $Z_t \sim N(0,1)$. The second term represents the random shocks to process V_t. The solution of SDE between s and t is given by $s < t$ (see Bouasabah, M., & Bensouda, C., 2017).

$$V_t = V_s e^{-\kappa(t-s)} + \mu\left(1 - e^{-\kappa(t-s)}\right) + \sigma e^{-\kappa t}\int_0^t e^{-\kappa u} dW_u$$

V_t is a unique solution of Vasicek model with $V_t \sim N\left(\mu + (V_0 - \mu)e^{-\kappa t}, \frac{\sigma^2}{2}(1 - e^{-2\kappa t})\right)$

The Euler discretization schema is given as $V_t = V_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t}) + \sigma\sqrt{\frac{1-e^{-2\kappa t}}{2\kappa}}\; Z_i$

where: $V_0 e^{-\kappa t} + \mu(1 - e^{-\kappa t})$ is the mean, Z_i is a one-dimensional noise process, and $\sigma\sqrt{\frac{1-e^{-2\kappa t}}{2\kappa}}$ is the volatility of the noise process.

2.2. The Fractional Brownian Motion

Fractional Brownian motion (fBm), which includes the long-range dependence, self-similarity, and depends on the Hurst parameter $H \in (0,1)$, is a zero-mean and continuous Gaussian process. The covariance function is given as

$$E\big(B_H(t), B_H(s)\big) = \frac{\sigma_H^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

$$\sigma_H^2 = \frac{1}{\big(\Gamma(H+1/2)\big)^2}\left[\int_0^{\infty}\left((1+s)^{H-1/2} - s^{H-1/2}\right)^2 ds + 1/2H\right] = \frac{1}{\Gamma(2H+1)\sin(\pi H)}$$

For $s = t$, $E\left[B_H^2(t)\right] = t^{2H}$. $E\left[\big(B_H(t) - B_H(s)\big)^2\right] = \sigma^2|t-s|^{2H}$ for $\forall s, t \in R$, $\sigma > 0$.

Fractional Brownian motion was firstly described by Mandelbrot and Van Ness (1968) as,

$$B_H(t) = \frac{1}{\Gamma(H+1/2)}\left(\int_{-\infty}^{0}\left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right) dB(s) + \int_0^t (t-s)^{H-\frac{1}{2}} dB(s)\right)$$

where : $\Gamma(z) = \int_0^{\infty} x^{z-1}e^{-x}dx$ represents the gamma function, $B(t)$ is a standard Brownian motion (for details, see Samorodnitsky and Taqqu, 1994). (fBm) may be seen as a weighted average of Gaussian white noise. Parameter H controls the regularity of the process. When $H > 1/2$, the process shows long range dependence and when $H \neq 1/2$, fBm is not semimartingale. In this case, Ito calculus is not available for this process. fBm has a self-similar property, so that for a constant $c > 0$, $B_H(ct) = c^H B_H(t)$.

A geometric fractional Brownian motion (GFBM) describes following SDE,

$$dX_t = \mu X_t dt + \sigma X_t dW_t^H$$

where: μ drift and σ volotility parameter are constants. W_t^H is fractional Brownian motion. The solution of (GFBM) is given by,

$$X_t = X_0 exp\left(\mu t - \frac{1}{2}\sigma^2 t^{2H} + \sigma W_t^H\right)$$

Parameter Estimation for the (GFBM) Model. Given the logarithmic returns $r_i = ln(X_i) - ln(X_{i-1})$, $i = 1,2,\dots,n$, then the sample mean m and sample variance v is calculated, respectively as,

$$m = \frac{1}{n}\sum_{i=1}^{n} r_i \quad \text{and} \quad v = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_i - \mu)^2}$$

Then, the parameters of (GFBM), μ and σ, are estimated. The drift μ and volatility σ parameters of (GFBM) model are estimated as follows (Feng, Z., 2018; Ibrahim, S. N. I., Misiran, M., & Laham, M. F., 2021).

$$\hat{\sigma} = \frac{v}{\sqrt{|\Delta t|^{2H}}} \quad \text{and} \quad \hat{\mu} = \frac{m}{\Delta t} + \frac{\hat{\sigma}^2}{2}$$

fBm is not stationary, however, its increments show a stationary process, thus, generally, inorder to make forecasts its increments are used.

$$E[|W_t^H - W_s^H|^2] = [|W_{t-s}^H|^2] = |t-s|^{2H}$$

(fBm) has the following scaling property,

$$E[|B_{t+\Delta}^H - B_t^H|^q] = K_q \Delta^{Hq}$$

$$K_q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^q e^{-x^2/2} dx$$

2.3 The Fractional Ornstein–Uhlenbeck Process(fOU)

The model(fOU) satisfies the following stochastic differential equation (SDE)

$$dX_t = \kappa(\mu - X_t)dt + \sigma dB_H(t)$$

The solution of SDE above is given by

$$X_{i\Delta} = \mu + \left(X_{(i-1)\Delta} - \mu\right)e^{-\kappa\Delta} + \sigma \int_{(i-1)\Delta}^{i\Delta} e^{-\kappa(i\Delta - s)} dB_H(s)$$

$$= e^{-\kappa\Delta} X_{(i-1)\Delta} + \mu(1 - e^{-\kappa\Delta}) + \sigma e^{-\kappa\Delta} \int_{(i-1)\Delta}^{i\Delta} e^{\kappa s} dB_H(s)$$

$$= e^{-\kappa\Delta}\left(X_{(i-1)\Delta} + \mu e^{\kappa\Delta} + B_H\left(\frac{\sigma^2}{2\kappa}(e^{2\kappa t} - 1)\right)\right)$$

To estimate the parameters of the fOU model, we use an estimation method proposed by Xiao *et al*. (2018). Let $X_{i\Delta}$, $i = 0,1, \dots,$, represent the discrete sample observations, then the model parameters are estimated. First, the Hurst exponent H is estimated as,

$$\hat{H} = \frac{1}{2} log_2 \left(\frac{\sum_{i=1}^{n-4} \left| X_{(i+4)\Delta} - 2X_{(i+2)\Delta} + X_{i\Delta} \right|^2}{\sum_{i=2}^{n-2} \left| X_{(i+2)\Delta} - 2X_{(i+1)\Delta} + X_{i\Delta} \right|^2} \right)$$

Then, using this estimated H parameter, the other parameters of the model are estimated as follows by the method of moments.

$$\hat{\sigma} = \left(\frac{\sum_{i=1}^{n-2} \left| X_{(i+2)\Delta} - 2X_{(i+1)\Delta} + X_{i\Delta} \right|^2}{n\left(4 - 2^{2\hat{H}}\right)\Delta^{2\hat{H}}} \right)^{1/2}$$

$$\hat{\mu} = \frac{(X_n - X_0)\sum_{i=0}^n X_i^2 - (\sum_{i=0}^n X_i(X_{i+1} - X_i))(\sum_{i=0}^n X_i)}{(X_n - X_0)(\sum_{i=0}^n X_i) - n(\sum_{i=0}^n X_i(X_{i+1} - X_i))}$$

$$\hat{\kappa} = \left(\frac{n \sum_{i=0}^{n} X_{i\Delta}^2 - (\sum_{i=0}^{n} X_{i\Delta})^2}{n^2 \hat{\sigma}^2 \hat{H} \Gamma(2\hat{H})} \right)^{-1/(2\hat{H})}$$

The Hurst exponent $H \in (0,1)$ characterizes the scaling behavior of the range of cumulative departures of a time series from its mean (Hurst, E., 1951; Mandelbrot and Wallis, 1968).

3. Volatility Modelling

The volatility, σ, is used in finance as a measure of variability (riskiness) in asset prices and it is not directly observable. It has an important role in asset/derivative pricing, risk management, and portfolio analysis. Estimating volatility accurately is valuable for both empirical and theoretical studies in finance. The implied volatility is a good indicator of the "fear" in the market. The volatility is generally estimated in two ways, namely the historical realized volatility and the implied volatility. Realized volatility is calculated based on observed historical prices, while the implied volatility is obtained from the market prices of financial derivatives. In this study, we also use the realized volatility, which is calculated from high-frequency data. Let S_t be the price of S&P500 index at time t. The process $S = (S_t)_{t \geq 0}$, $t \in [0, T]$ satisfies the following stochastic differential equation (SDE)

$$S_t = S_0 + \int_0^t \sigma_s dW_s$$

The volatility only might be derived from the observable proxies of the realized variance. Let be S_t, over time grid $t_0 < t_1 < \cdots < t_n = T$,using the most recent m observations on the S_{t_i} . We consider the price S_t, on time grids $t_0 < t_1 < \cdots < t_n = T$. Using the most recent m observations on the S_{t_i} is deduced the annual realized variance as

$$\text{Annualized Realized Volatility} = RV_n^{(m)} = 100 \times \sqrt{\frac{252}{m} \sum_{i=0}^{m-1} \left(ln \left(\frac{S_{t_n - t_i}}{S_{t_n - t_i + 1}} \right) \right)^2}$$

$RV_n^{(m)}$ is the annualised realized volatility for S&P500 on day t. For $|t_i - t_{i-1}| \to 0$, RV_n approaches to the quadratic variance, thus the realized variance is used as an unbiased estimator of integrated variance (Guo, I., Loeper, G., Obłój, J., & Wang, S., 2020).

$$RV_n^{(m)} \to \frac{100^2}{T - t_0} \int_{t_0}^{T} \sigma_u^2 du$$

The realized volatility measures the annualized standard deviation in the daily price return of an index over a given period. Realized variance is used as an unbiased estimator of integrated variance,

$$RV_n^{(m)} \sim IV_t$$

We need to find a proxy for spot volatility. In general, considered volatility proxies are computed at a daily frequency. Then, we can estimate the integrated variance for a step size $\Delta > 0, \mathrm{n} \in \mathrm{N}$,

$$IV_t = \int_{t-\Delta}^{t} \sigma_u^2 du \quad , \quad t = \Delta, 2\Delta, \dots, n\Delta$$

The value of expected variance, $E_t\left[\int_t^T \sigma_s^2 ds\right]$ can be computed from the price of options with an expiry T. We obtain a proxy for spot volatility as $\hat{\sigma}_t^2 = \frac{1}{\Delta}\widehat{IV}_t + \epsilon_t$, where ϵ_t are i.i.d centered random variables. The VIX is used as an implied volatility estimate proxy.

$$VIX_t = \sqrt{\frac{1}{\Delta} E_t\left[\int_t^{t+\Delta} \sigma_s^2 ds\right]}$$

Another parametrization is

$$VIX_t = \sqrt{price_t^{mkt}\left(-\frac{2}{\Delta} ln\left(\frac{S_{t+\Delta}}{S_t}\right)\right)}$$

where: $\Delta = 30$ days and $price_t^{mkt}$ is the market price at the time t. We need to derive a proxy for the volatility process from index prices, since we cannot directly observe the volatility. The volatility proxies can be computed in different ways. Let H_t is the highest price, L_t is the lowest price, O_t is the opening price and C_t is the closing price of day t. The proxies of daily volatility, based on the highest and lowest, opening and closing prices of the day are:

Parkinson (1980) volatility: $\sigma_n = \sqrt{\frac{1}{4\ln 2}\frac{252}{n}\sum_{k=1}^{n}\left(ln\frac{H_k}{L_k}\right)^2}$

German-Klass Volatility: $\sigma_{GK} = \sqrt{\frac{252}{n}\sum_{t=1}^{n}\left[\frac{1}{2}\left(ln\frac{H_t}{L_t}\right)^2 - (2ln2-1)\left(ln\frac{C_t}{O_t}\right)^2\right]}$

Exponentially-weighted Moving Average (EWMA)

Current volatility estimate σ_t uses m most recent observed returns $r_{t-m+1}, \dots, r_t$. The decay factor λ is a known parameter.

$$\sigma_t^2 = \frac{1-\lambda}{1-\lambda^m}\sum_{i=0}^{m} \lambda^{m-i} r_{t-m+i}^2$$

Annualized volatility: $\sigma_t = \sqrt{\frac{252(1-\lambda)}{1-\lambda^m}\sum_{i=0}^{m} \lambda^{m-i} r_{t-m+i}^2}$

The above methods are based on daily prices. Besides, all of these methods have similar results.

3.1 Rough Volatility Models

Measure of Roughness of the Log-volatility

In this section, we consider the scaling of the moments of the increments of the log-volatility. Gatheral (2018) propose fitting the empirical absolute moment of order q, for different q values, as

$$log\left[\frac{1}{n}\sum_{k=1}^{n}|log(\sigma_{t+\Delta})-log(\sigma_t)|^q\right]\approx qHlog(\Delta)+\eta_q$$

Then, the empirical estimates of H are determined as $1/q$ times the slope (qH) of the above linear regression. The rough volatility models have the paths with fBM with $H\in(0,1/2)$, which are rougher than trajectories of a standart Brownian motion. For very long mean reversion time $\kappa\ll 1/T$ and $H\in(0,1/2)$ log-volatility process σ_t follows locally as an FBM process (Gatheral ***et al.***, 2018, and Bennedsen ***et al.***, 2016). Roughness of volatility is closely related to the Hurst parameter, H. As H approaches to 0, the paths become more irregular. An fBm satisfies the relationship $\mathrm{H}=\alpha+1/2$, which means that $\mathrm{H}<1/2$ implies roughness. Bennedsen ***et al***. (2016) propose the following relationship,

$$1-Corr(log\sigma_t\,,log\sigma_{t+\Delta})\sim c|\Delta|^{2\alpha+1}\ ,\qquad |\Delta|\to 0$$

where: c is a constant, ρ is the autocorrelation function of log volatility, $|\Delta|$ is the lag time, $\alpha\in\left(-\frac{1}{2},\infty\right)$ and α is called the roughness index of volatility. We estimate the volatility of the S&P over days. We study the scaling measure as follows,

$$m(\Delta,q)=E[|log(\sigma_{t+\Delta})-log(\sigma_t)|^q]$$

For various Δ and q, the smallest Δ is one day.

$$m(\Delta,q)\sim c|\Delta|^{2\alpha+1}\ ,\qquad |\Delta|\to 0$$

$$logm(\Delta,q)=b+alog|\Delta|+\epsilon_\Delta$$

where: $\Delta=5,10,15,20,25,30,35,40,45,50$, $q=2$. The relationship $a=2\alpha+1$ allows us to estimate α using $\hat{\alpha}=\frac{\hat{a}-1}{2}$.

3.2 The Fractional Ornstein-Uhlenbeck Volatility Model(fOUVM)

The log-realized volatility behaves similarly to fBm with $H<1/2$. But, $\sigma_t=\sigma_0 exp(vB_t^H)$ model, so the expected volatility $E(\sigma_t)=\sigma_0 exp\left(\frac{v^2t^{2H}}{2}\right)$ is not stationary, because $B_t^H=N(0,t^{2H})$. σ_0 is the average realized volatility and using a fractional Ornstein-Uhlenbeck

$$dlog(\sigma_t)=\kappa\big(\mu-log(\sigma_t)\big)dt+\vartheta dB_t^H\quad ,\quad t\in[0,T]$$

where: $\mu\in R$, $\vartheta>0$. The solution of the fOU proces is given by,

$$log(\sigma_t)=\mu+\vartheta\int_{-\infty}^{t}e^{-\kappa(t-s)}dB_s^H$$

If $\kappa\ll 1/T$, then log-volatility process behaves locally as fBm. The variance forecast formula is,

$$(log\sigma_{t+\Delta}^2|\mathcal{F}_t) = \frac{cos(\pi H)}{\pi}\Delta^{H+1/2}\int_{-\infty}^{t}\frac{log\sigma_s^2}{(t-s+\Delta)(t-s)^{H+1/2}}ds$$

3.2.1 The Rough Fractional Stochatic Volatility (RFSV) Model

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t$$

$$\sigma_t = exp(X_t) \quad \text{and} \quad dX_t = \kappa(\mu - X_t)dt + \nu dB_t^H$$

where: κ is the parameter of mean-reversion of the instantaneous volatility.

The discritezation of (fOU) on grid time $t_1, t_2, \dots, t_n$ on $[0, T]$ is given as

$$\mathrm{X(t_{i+1}) = X(t_i) + \kappa\big(\mu - X(t_i)\big)(t_i - t_{i-1}) + \nu\big(W^H(t_i) - W^H(t_{i-1})\big)}$$

3.3 The Rough Bergomi Model

The rough Bergomi (rBergomi) model is introduced by Bayer, C., Friz, P., and Gatheral, J. (2016) as follows

$$\frac{dS_t}{S_t} = \sqrt{V_t}\left(\rho dW_r + \sqrt{1-\rho^2}dB_r\right)$$

$$V_t = \xi(t)exp\left(\eta W_t^H - \frac{1}{2}\eta^2 t^{2H}\right)$$

where: $\nu \approx 1.7$, $C_H = \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}$, and η is instantaneous volatility of the instantaneous variance (vol-of-vol) : $\eta = 2\nu\frac{C_H}{\sqrt{2H}} = 2\nu\sqrt{\frac{\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}$

Approximately $\eta \approx 2.5$, Riemann-Liouville fBm: $W^H(t) = \int_0^t \mathrm{K}^H(t-s)dZ_s \ ,0 \le s \le t$

$\mathrm{K}^H(t-s) = \sqrt{2H}(t-s)^{H-1/2}$, $\xi_0(\mathrm{t}) = (0.234)^2\sqrt{1+\mathrm{t}}$ and $\mathrm{t} \to \xi_0(\mathrm{t})$ forward variance curve, known at time 0.

The Hybrid Scheme (Bayer, C., Ben Hammouda, C., and Tempone, R., 2020).

$$\widetilde{W}_{\frac{i}{N}}^H = \sqrt{2H}\left(W_i^2 + \sum_{k=2}^{i}\left(\frac{b_k}{N}\right)^{H-1/2}\left(W_{\frac{i-(k-1)}{N}} - W_{\frac{i-1}{N}}\right)\right)$$

where: N is number of steps and W_i^2 is a Gaussian variable.

$$b_k = \left(\frac{k^{H+1/2} - (k-1)^{H+1/2}}{H+1/2}\right)^{1/(H-1/2)}$$

where: V_t is the instantaneous variance process, $\eta \in (0,1)$ is a parameter, $H \in (0, 1/2)$, $\xi(t)$ is the forward variance curve, W_t^H is a Riemann-Liouville fBm. B_r, W_r are independependent Brownian motions. A generalized volatility process σ_t is proposed by Merino, R., Pospíšil, J., Sobotka, T., Sottinen, T., and Vives, J. (2021) as follows

$$\sigma_t = \sigma_0 exp\left(\eta B_t^H - \frac{1}{2}\alpha\eta^2 r(t)\right) \;, t \geq 0$$

where: $\sigma_0 > 0$,$\alpha \in [0,1]$ and $H < 1/2$. η is the volatility of the volatility and calculated as

$$\Upsilon_\Delta = \frac{1}{(n-\Delta+1)}\sum_{i=\Delta}^{n}[log(\hat{\sigma}_i) - log(\hat{\sigma}_{i-\Delta})]^2$$

Finally, the following estimator is found (for details see, Garcin, M., and Grasselli, M., 2020).

$$\hat{\eta}_\Delta = \sqrt{2\left(\sqrt{1+4\widehat{\Upsilon}_\Delta} - 1\right)}$$

For, $\alpha = 0$, the RFSV model and $\alpha = 1$, we obtain the rough Bergomi model.

For every $\varepsilon > 0$,

$$r(t) = 2H\int_0^t (t-s+\varepsilon)ds = (t+\varepsilon)^{2H} - \varepsilon^{2H}$$

If $\varepsilon = 0$ then $r(t) = t^{2H}$ (the variance of the standard fBm)

3.3.1 Simulation of the Rough Bergomi Model

Jacquier *et al*. (2018) propose a simulation approach as follows. Let $\mathrm{t} = \mathrm{t_i} = \mathrm{i}\Delta$, $0 = 0,1,2, \dots \mathrm{n}$ represent the time grid.

1) Simulating the Volterra process $W^H = \left(W^H(t)\right)_{t\geq 0}$, $W^H(t) = \int_0^t (t-s)^{H-0.5} dZ_s$,

2) Simulating the variance process $\mathrm{V_t} = \xi_0(\mathrm{t})\mathcal{E}\left(2vC_H \int_0^{\mathrm{t}} (\mathrm{t}-\mathrm{s})^{\mathrm{H}-0.5}\mathrm{dZ_s}\right)$, $v \approx 1.2287$

$C_H = \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(H+1/2)\Gamma(2-2H)}}$ and Doléans-Dade exponential $\mathcal{E}(\Upsilon_\mathrm{t}) = \exp\left(\Upsilon_\mathrm{t} - \frac{1}{2}\mathrm{E}(\Upsilon_\mathrm{t}^2)\right)$

3) Set the paths of Brownian motion Z driven by X as follows,

$\mathrm{Z_{t_i}} = \mathrm{Z_{t_{i-1}}} + \mathrm{n}^{\mathrm{H}-0.5}\left(W^H(t_i) - W^H(t_{i-1})\right)$ and $\bar{\mathrm{Z}}_{\mathrm{i}-1} = \mathrm{n}^{\mathrm{H}-0.5}\left(W^H(t_i) - W^H(t_{i-1})\right)$

Let $Z_i^1 \sim N(0, 1/n)$ and $\mathrm{W^H(t_i)} - \mathrm{W^H(t_{i-1})} = \rho\bar{\mathrm{Z}}_{\mathrm{i}-1} + \sqrt{1-\rho^2}\mathrm{Z_i^1}$

4) Simulating the S&P500 index price $S(t_i) = exp\left(X(t_i)\right)$

$$X(t_{i+1}) = X(t_i) - \frac{1}{2}V(t_i)\Delta t_i + \sqrt{V(t_i)}\left(W^H(t_i) - W^H(t_{i-1})\right)$$

$$\Delta t_i = t_i - t_{i-1}$$

4. Application

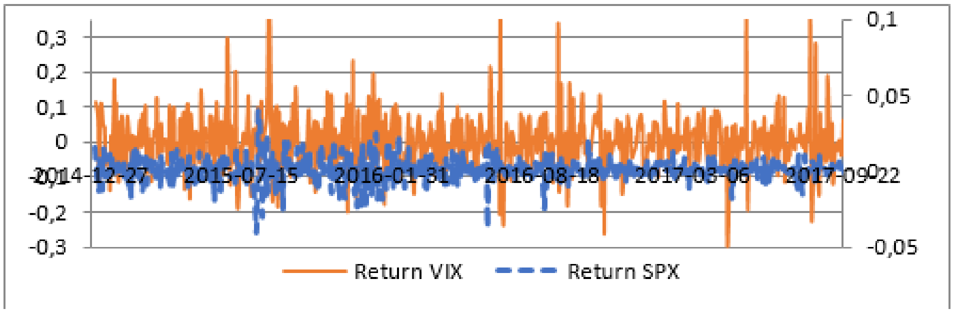
Our data set consists of closing prices of the S&P500 and the VIX Indices, daily from 01.01.2015 to 01.01.2020, retrieved from Yahoo Finance. In this section, we use daily price data of S&P 500 and VIX indices. We apply our models to VIX index daily data spot volatility and the implied volatility of S&P500. The data are obtained from CBOE from 01.01.2015 to 01.01.2020. (http://www.cboe.com/products/vix-index-volatility/vix-options). In this study, we take $\Delta = 1/252$ and $T = 4$ years. There is a total of 1256 observed data. For the S&P500 index, we use proxy daily spot variances by daily realized variance estimates from the Oxford-Man Institute of Quantitative Finance Realized Library, (https://realized.oxford-man.ox.ac.uk/data/download). The VIX is a volatility index on the Chicago Board of Exchange (CBOE) and is the measure of the market's expectation of the 30-day volatility implied by the S&P 500 (CBOE, 2009). The VIX is a weighted average of implied volatilities of options on the S&P 500 Index with various strikes and maturities. We compare the VIX index with historical volatility. The observed data are used to estimate realized volatility. The roughness of the realized volatility is assessed by estimating the Hurst parameter.

Figure 1 compares the VIX and the S&P500 indices over the all-observation interval. A spike in the VIX is accompanied by a simultaneous negative reaction in SPX index prices. Between VIX and SPX indices, a negative correlation exists. When the S&P500 index suffer sharp declines, the VIX volatility index tends to rise.

The VIX index measures the expected volatility of the S&P500 Index (SPX) over the next 30 days. The VIX it goes higher the higher trader's expectations are for the short term market volatility.

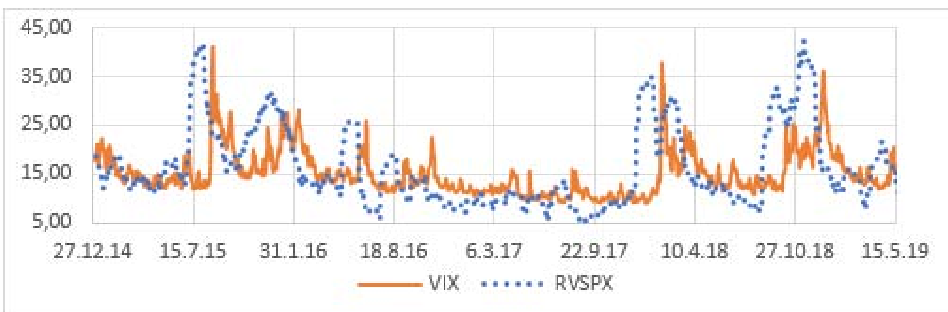
Figure 1. S&P 500 and VIX Levels

Figure 2. Log Returns of VIX and S&P 500 Indices



We analysed the daily log returns of the VIX and S&P 500 indices. Figure 2 shows two indices representing the volatility clusters in their log-returns. In other words, the large movements tend to be followed by large movements and vice versa.

Figure 3. The VIX and the Realized Volatility of the S&P 500 One-month Ahead

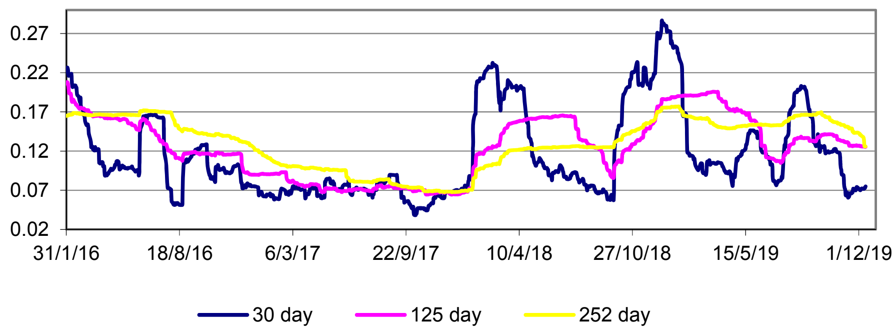


The realized (historical) volatility can be expressed as a measure of the past changes in stock prices. Implied volatility denotes the expected future rate of change of index prices. In Figure 3, when comparing the implied volatility (measured by the VIX) with the realized volatility of the S&P 500 index, the VIX generally appears higher than the realized variance. This difference between two variables may be considered as a risk premium. Thus, the VIX index can be used as a proxy for the realized volatility. Furthermore, it may give early signals about future changes in the S&P500 index.

RiskMetrics suggests to use the decay factor λ of 0.94 and 0.97 for daily and monthly data, respectively. To calculate the historical spot volatility, we use the Exponential Weighted Moving Average (EWMA) with smoothing parameter $\lambda = 0.94$ and r is the return of SPX.

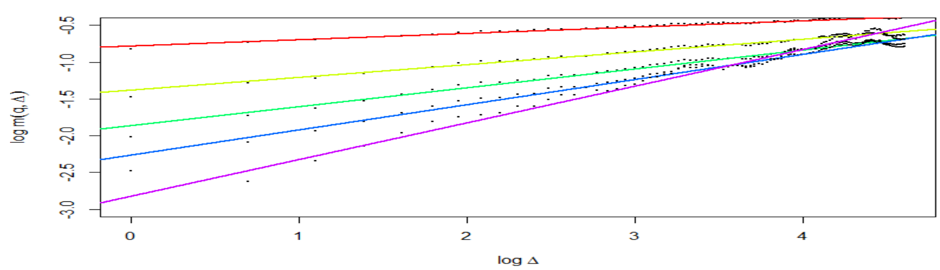
$$\sigma_{EWMA}(t+1) = \sqrt{\lambda.\sigma^2_{EWMA}(t) + (1-\lambda).r^2(t)}$$

Figure 4. SPX (EWMA) Volatility



The EWMA method is easy to calculate and gives us a low RMSE. The VIX levels and S&P500 realized volatility levels, calculated using closing price levels, are given for each trading day.

Figure 5. $\mathbf{Log(m(q,\Delta))}$ **as a Function of** $\mathbf{Log(\Delta)}$**, S&P500,** $\mathbf{q} \in$



$\{\mathbf{0.5, 1, 1.5, 2, 3}\}$

Here, the volatility proxies used are the precomputed 5-minute realized variance estimates by the Oxford-Man Institute of Quantitative Finance. Figure 5 shows the estimates of the volatility process smoothness of SPX. The volatility process is not directly observable from the market. So, an exact computation of $m(q,\Delta)$) is not possible in practice. We must derive therefore proxy spot volatility values by using appropriate methods.

Figure 6 shows that the volatility of S&P 500, generated by the Fractional Rough Ornstein-Uhlenbeck Volatility Model (FROUSV) shows very similar behavior to one day ahead VIX Index level. The (FROUSV) model is consistent with the mean reversion property of volatility. There is a significant degree of variation.

Figure 6. VIX and the Estimated Volatility from Model for SPX

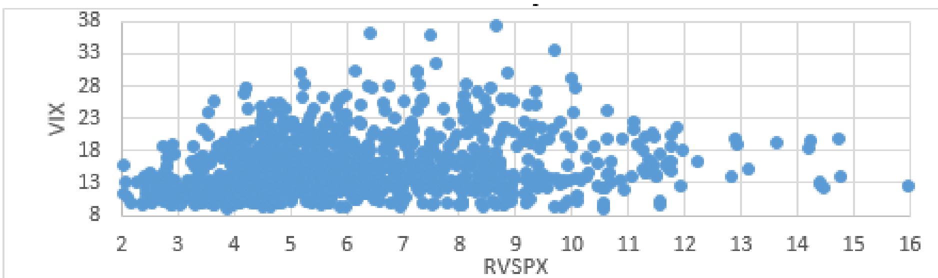


Table 1. Descriptive Statistics of S&P500 and VIX

Index	Mean	Std.dev.	Kurt.	Skew.	Min	Max	J. B
S&P500	2465.22	364.42	-1.2	0.2	1829.08	3329.62	92.65
VIX	15.06	4.23	3.49	1.55	9.14	40.74	987.95

Table 1 summarizes the moments of S&P500 and VIX indices. It shows that the VIX index is much more volatile than the S&P500 index. In addition, The S&P500 index has positive skewness and weak kurtosis. The VIX returns have positive skewness and strong kurtosis (>3). The Jarque Bera statistics show that both of them do not satisfy the normality assumption. The calibration of parameters of (FROUSV) model process to $X_t = \log(\mathrm{VIX_t})$ observation data is given in Table 2.

For the VIX index, instantaneous volatility values can be derived based on the (FROUSV) model, given by

$$V_{t+1} = V_t(1 + 0.0154(2.6834 - V_t)(0.004) + (0.001)(0.06186)(0.004)^{0.42}Z_1 + (0.05433)(0.004)^{2.(0.42)}Z_2$$

Tabel 2. Model Parameter Estimates

	$\hat{\kappa}$	$\hat{\mu}$	$\hat{\sigma}$	$\hat{H}$	$\hat{\eta}$	$\hat{\lambda}$	$\hat{\alpha}$
SPX(RV)	0.0077	9.25	1.325	0,23			-0.27
Log(VIX)	0.0154	2.68	0.062	0.33	0.001	0.0543	-0.17
Log(SPX)	0.1988	8.26	0.134	0.23	0,001	0.0137	-0.27
EWMA(SPX)	0.0188	-0.0005	0.067	0.54	0.001		0.043

The estimated Hurst parameter $\mathrm{H} = 0.33$ of the realized volatility process is in the rough volatility range, $0 < H < 0.5$. Therefore, we can model the VIX index using a rough volatility model. Furthermore, the value of $\hat{\kappa}$ is higher than$1/T$ value. This result represents the roughness of the VIX index.

Figure 7. Real VIX and VIX Estimates Based on the Model (MAPE=0.1094)

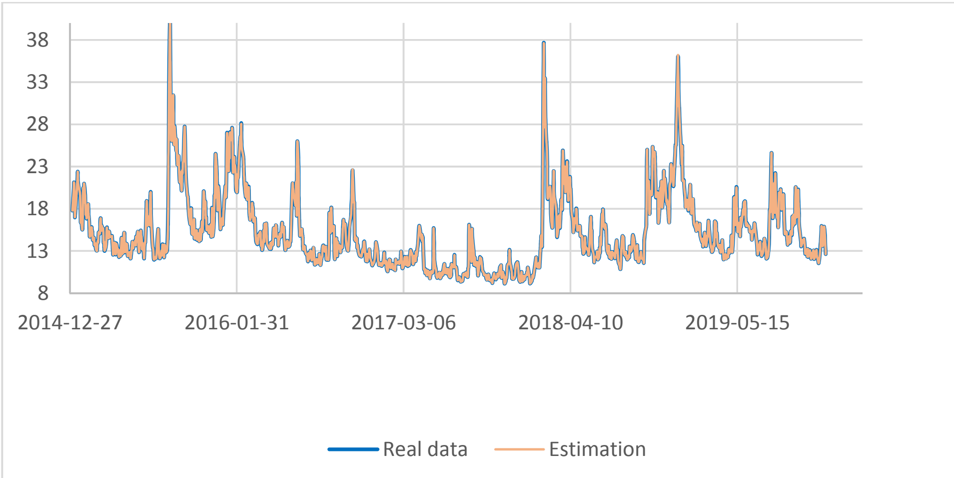


Figure 8. Real and Estimated Values from Model, for S&P500 Index Prices (MAPE=0.1094)

S&P500 index prices can be simulated using the following stochastic equation

$$S_{t+1} = S_t(1 + 0.1988(8.2616 - S_t)(0.004) + (0.001)(0.1338)(0.004)^{0.181}Z_1 + (0.01368)(0.004)^{2.(0.181)}Z_2$$

In Figure 8, it is observed that the simulated and actual graphs of the S&P500 index prices look very much alike.

5. Conclusions

In this paper, we investigate the joint modelling of S&P500 and VIX indices using the rough Bergomi volatility approach. The log-volatility is modelled using a Rough Fractional Ornstein-Uhlenbeck process (RFOU). The proxy volatility process for the SPX index is calculated using the Exponential Weighted Moving Average (EWMA) method. The daily realized variance estimates, which are proxies for daily spot (squared) volatilities of S&P500, and the daily VIX index data are calibrated in our model. The VIX index contains some predictive information about future SPX volatility. We found that by using a monofractal scaling relationship the log-volatility might be modelled with a rough fractional Ornstein-Uhlenbeck process and the Hurst exponent H is below 0.5 for daily time scale. Furthermore, the rough volatility models may capture the dynamics of historical and implied volatilities. The variance process is not a Markov process nor semimartingale, the scaled VIX index can be used as a proxy for the implied volatility process. The realized variance process imitates the fBm behavior very well. The VIX index can capture the future volatility of the SPX, and hence it can be used to predict the future movements of the S&P500. The VIX volatility index is a proxy for implied volatility.

Finally, we derive the rough implied volatility process and make new predictions for the S&P500 price dynamics based on a Rough Fractional Ornstein-Uhlenbeck model with rough volatility. The stochastic volatility process can also be modelled as a Volterra process (*i.e.*, a moving average of a standard Brownian motion). In the rough volatility framework, using the form of the integration kernel in the moving average is very important.

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