8. Modeling Heavy-Tailed Stock Index Returns Using the Generalized Hyperbolic Distribution

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Abstract

In the present study, we estimate the parameters of the Generalized Hyperbolic Distribution for a series of stock index returns including the Romanian BETC and indexes from other two Eastern European countries, Hungary and the Czech Republic. Using different econometric techniques, we investigate whether the estimated Generalized Hyperbolic Distribution is an appropriate approximation for the empirical distribution computed by non-parametric kernel econometric methods. The main finding of the analysis is that the probability density function of the estimated Generalized Hyperbolic Distribution represents a very close approximation (at least up to the 4th order term) of the empirical probability distribution function.

Keywords: Generalized Hyperbolic Distribution, heavy-tailed returns, non-parametric density estimation

JEL Classification: C13, C14, C16, G10

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1. Introduction

It is widely known that the assumption according to which the financial assets returns are normally distributed is not supported by empirical evidence. Cont (2001) concludes that the precise form of the tail of financial returns` distribution is difficult to determine, and that in order for a parametric distributional model to reproduce the properties of the empirical distribution it must have at least four parameters: a location parameter, a scale parameter, a parameter describing the decay of the tails and an asymmetry parameter. Therefore, it is important to develop theoretical models based

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on other distribution classes, explaining asymmetry and heavy tail phenomena. In this sense, stable distributions and normal mixtures distributions have been used with considerable success. In recent years, more realistic stochastic models for price movements in financial markets have been developed by replacing the classical Brownian motion with Levy processes.

The Generalized Hyperbolic Levy processes turned out to provide an excellent fit to observed market data. Many authors have successfully fitted Generalized Hyperbolic Distributions and, in particular, Normal Inverse Gaussian laws to returns in financial time series [Eberlein and Keller, 1995; Prause, 1997; Barndorff-Nielsen, 1997; Prause, 2000; Barndorff-Nielsen and Shephard, 2001, 2005; Schoutens, 2003].

This has encouraged modeling the time dynamics of financial markets by stochastic processes using Generalized Hyperbolic or Normal Inverse Gaussian laws and associated Levy processes as building blocks [Rydberg, 1997; Bibby and Sorensen, 1997; Rydberg, 1999; Prause, 1999; Raible, 2000; Barndorff-Nielsen and Shephard, 2001, 2005; Barndorff-Nielsen, 2001; Eberlein, 2001; Schoutens, 2003; Cont and Tankov, 2004].

The class of Generalized Hyperbolic Distributions includes the standard hyperbolic distribution, the normal inverse Gaussian distribution, the scaled t-distribution and the variance-gamma distribution. The scaled t-distribution was used in finance by Praetz (1972) and Blattberg and Gonedes (1974), while Madan and Seneta (1990) introduced the variance-gamma distribution in the financial literature. The tail behavior of the Generalized Hyperbolic Distributions ranges from Gaussian tails via exponential tails to the t-distribution power tails.

In the present study we estimate the parameters of the Generalized Hyperbolic Distribution for a series of stock index returns including the Romanian BETC and indices from other two Eastern European countries, Hungary and the Czech Republic. Using different econometric techniques, we investigate whether the estimated Generalized Hyperbolic Distribution is an appropriate approximation for the empirical distribution computed by non-parametric kernel methods.

The paper is organized as follows: in the second section we present the main properties of the Generalized Hyperbolic Distribution, in the third section we present the data and the methodology employed for the analysis, in the fourth section we estimate econometrically the parameters of the Generalized Hyperbolic Distribution and compare the estimated distribution to the empirical one, and the final section includes conclusions.

2. The Generalized Hyperbolic Distribution

The Generalized Hyperbolic Distribution was introduced by Barndorff-Nielsen (1977). The generalized hyperbolic distribution has five parameters. If the random variable X follows a Generalized Hyperbolic Distribution one can write

$$X \sim GH(\lambda, \alpha, \beta, \delta, \mu)$$

where: μ is a location parameter, δ serves for scaling, α determines the shape, β determines the skewness, and λ influences the kurtosis and characterizes the classification of the Generalized Hyperbolic Distributions.

The probability density function of the Generalized Hyperbolic Distribution is

$$\rho_{GH}\left(x;\lambda,\alpha,\beta,\delta,\mu\right)=a\left(\lambda,\alpha,\beta,\delta,\mu\right)\left(\delta^2+\left(x-\mu\right)^2\right)^{1/2\lambda-1/4}\cdot$$
$$\cdot B\left(\lambda-0.5,\alpha\sqrt{\delta^2+x^2-2x\mu+\mu^2}\right)e^{\beta(x-\mu)}$$

where:

$$a\left(\lambda,\alpha,\beta,\delta,\mu\right)=\frac{\left(\alpha^2-\beta^2\right)^{1/2\lambda}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda}B\left(\lambda,\delta\sqrt{\alpha^2-\beta^2}\right)}$$

and $B(\lambda,\cdot)$ denotes the modified Bessel function of the third kind with index λ.

There are two subclasses of the Generalized Hyperbolic Distribution that are extensively used in the finance theory. The first is obtained in case that $\lambda=1$ and is called the (simple) Hyperbolic Distribution $H(\alpha,\beta,\delta,\mu)$:

$$\rho_H\left(x;\alpha,\beta,\delta,\mu\right)=\frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta B\left(1,\delta\sqrt{\alpha^2-\beta^2}\right)}\exp\left(-\alpha\sqrt{\delta^2+\left(x-\mu\right)^2}+\beta\left(x-\mu\right)\right)$$

The name of Hyperbolic Distribution derives from the fact that the log-pdf represents the equation of a hyperbola. In the case of the Gaussian distribution the log-pdf represents the equation of a parabola.

In case that $\lambda=-1/2$ one obtains the second important subclass, namely the Normal Inverse Gaussian Distribution $NIG(\alpha,\beta,\delta,\mu)$:

$$\rho_{NIG}\left(x;\alpha,\beta,\delta,\mu\right)=\frac{\alpha\delta}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta\left(x-\mu\right)\right)\frac{B\left(1,\alpha\sqrt{\delta^2+\left(x-\mu\right)^2}\right)}{\sqrt{\delta^2+\left(x-\mu\right)^2}}$$

The NIG Distribution is the only subclass of the GH Distribution that is closed to convolutions, which is an appealing property for modeling financial returns. More specifically, if $X_1\sim NIG(\alpha,\beta,\delta_1,\mu_1)$ and $X_2\sim NIG(\alpha,\beta,\delta_2,\mu_2)$ then the distribution of the sum of the two random variables has also a NIG distribution $X_1+X_2\sim NIG(\alpha,\beta,\delta_1+\delta_1,\mu_1+\mu_2)$.

An important property is that the Generalized Hyperbolic Distribution is a normal variance-mean mixture where the mixing distribution is a generalized inverse Gaussian distribution $GIG(\lambda,\delta,\gamma)$, a class of distributions that generalizes the Gamma distribution. More precisely, if the conditional distribution of a random variable is given by $X\,|\,\sigma^2\sim N(\mu+\beta\sigma^2,\sigma^2)$ and the variance is distributed $\sigma^2\sim GIG(\lambda,\delta,\gamma)$, then $X\sim GH\left(\lambda,\sqrt{\beta^2+\gamma^2},\beta,\delta,\mu\right)$. Therefore, the models based on the Generalized Hyperbolic Distribution are a generalization of the normal mixture models with a discrete mixing distribution (Alexander and Lazar, 2006).

In recent years, more realistic stochastic models for price movements in financial markets have been developed by replacing the classical Brownian motion with Levy processes (i.e., processes with stationary and independent increments). At present, the research regarding the modeling of financial assets is focused on the market model based on the Generalized Hyperbolic Distribution. The Generalized Hyperbolic process L_t is a Levy process such that $L_1 \sim GH(\lambda, \alpha, \beta, \delta, \mu)$. It is known that a Levy process can be decomposed into a deterministic trend, a diffusion based on the Brownian motion, and a jump process. A characteristic of the Generalized Hyperbolic process refers to the fact that it has no diffusion component. Therefore, it is a "pure jump" process.

Since the market model based on the exponential Levy motion is not complete, the risk neutral measure, which is used for pricing contingent claims, is not unique. An important issue in this context is the usage of the Esscher transform to determine a market neutral measure characterized by the fact that the price of the financial asset is an exponential Levy process under this measurement. An appealing property for financial modeling is that the Generalized Hyperbolic process is closed to the Esscher transform. More specifically, if L_t is a $GH(\lambda, \alpha, \beta, \delta, \mu)$ process with respect to the market measure, after applying the Esscher transform on parameter θ, it becomes a $GH(\lambda, \alpha, \beta + \theta, \delta, \mu)$ process with respect to the risk neutral measure.

Another important characteristic is that the Generalized Hyperbolic process can be considered a Brownian motion with drift that evolves according to an "operational time" instead of the physical time. More specifically, the Generalized Hyperbolic process can be written $L_t = a\tau(t) + W_{\tau(t)}$, where W_t is a Brownian motion, and the "operational time" $\tau(t)$ is an increasing Levy process (i.e. a subordinator). The concept of "operational time" was introduced into the financial literature by Praetz (1972). Clark (1973) and Epps (1973) linked the "operational time" to the trading volume.

The properties presented above imply that the Generalized Hyperbolic process is an appealing process to model the financial returns. In the next sections of the study we analyze whether the econometrically estimated Generalized Hyperbolic Distribution is an appropriate approximation for the empirical distribution in case of stock index returns.

3. Data and methodology

The data used in the study consists of daily returns between January 1998 and September 2008 for ten stock indexes both from developed countries and emerging economies: USA (SP500), Japan (Nikkei 225), Hong Kong (Hang Seng), Germany (DAX), UK (FTSE100), France (CAC40), Spain (IBEX35), the Czech Republic (PX50), Hungary (BUX) and Romania (BETC).

The skewness and the kurtosis of the returns, together with the Kolmogorov-Smirnov and the Anderson-Darling normality test statistics are presented in Table 1.

Table 1

The distributional characteristics of the returns

Index	Skewness	Kurtosis	KS statistic	AD statistic
SP500	-0.2036	6.6866	0.0541	4.4626
Nikkei 225	-0.1085	4.5880	0.0417	2.7928
Hang Seng	0.1590	8.4344	0.0691	∞
DAX	-0.1399	5.5697	0.0617	4.5234
FTSE100	-0.0450	5.9987	0.0563	∞
CAC40	-0.0636	5.9566	0.0551	4.3339
IBEX35	-0.1350	6.0346	0.0571	4.5400
PX50	-0.0546	7.0230	0.0502	∞
BUX	-0.4920	11.8510	0.0544	∞
BETC	-0.3671	7.6766	0.0752	4.9500

The returns are characterized by heavy tails, both the Kolmogorov-Smirnov test and the Anderson-Darling test rejecting the null hypothesis that the returns are normally distributed.

The methodology for analyzing whether the Generalized Hyperbolic Distribution is an appropriate candidate for modeling stock index returns distribution consists of the following steps:

1. econometrically estimating the parameters of the Generalized Hyperbolic Distribution by the Maximum Likelihood Estimation (MLE) method;
2. computing the empirical distribution by non-parametric econometric techniques (the kernel density estimation methodology is presented in the Appendix);
3. comparing the first four estimated centered moments (i.e. mean, variance, skewness, kurtosis) to the empirical ones;
4. comparing the log-pdf of the estimated Generalized Hyperbolic Distribution and of the empirical distribution;
5. analyzing the q-q plots;
6. computing the distance between the empirical distribution and the estimated ones (the Generalized Hyperbolic Distribution and the benchmark Gaussian distribution) using the Kolmogorov-Smirnov and the Anderson-Darling statistics.

The econometric methods and techniques employed in the study are implemented in Octave, a Matlab-type free software.

4. Estimation results

The estimated parameters of the Generalized Hyperbolic Distribution for the ten return series analyzed in this study are presented in Table 2.

Table 2

The estimated parameters of the Generalized Hyperbolic Distribution

Index	λ	α	β	δ	μ
SP500	0.2718	95.08	-4.646	0.00585	0.00071
Nikkei 225	-1.0750	85.66	-4.790	0.02271	0.00089
Hang Seng	-0.0009	53.44	1.772	0.00755	-0.00029
DAX	-1.6180	49.13	-3.543	0.02376	0.00098
FTSE100	-1.4432	60.37	-1.249	0.01675	0.00016
CAC40	-1.2657	54.76	-1.499	0.01912	0.00041
IBEX35	-0.6741	66.91	-3.161	0.01528	0.00078
PX50	-1.1282	51.12	-1.021	0.01543	0.00052
BUX	-0.5087	35.27	-3.446	0.01007	0.00128
BETC	-1.0588	48.25	-6.022	0.01506	0.00191

The empirical centered moments and those computed by means of the estimated Generalized Hyperbolic Distribution are presented in Table 3.

Table 3

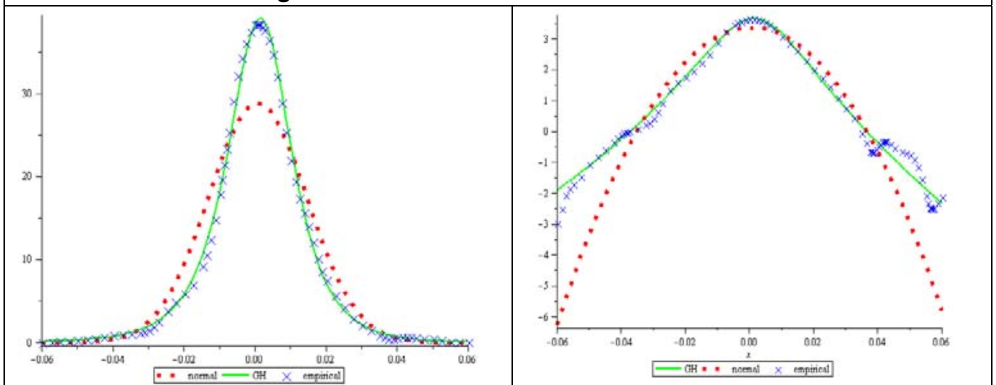
The empirical and the estimated centered moments of the returns

		mean	**variance**	**skewness**	**kurtosis**
SP500	empirical	0.000066	0.000140	-0.2036	6.6866
	estimated	0.000064	0.000140	-0.1996	6.6836
Nikkei 225	empirical	-0.000108	0.000209	-0.1085	4.5880
	estimated	-0.000108	0.000209	-0.1085	4.5866
Hang Seng	empirical	0.000196	0.000276	0.1590	8.4344
	estimated	0.000196	0.000276	0.1590	8.4348
DAX	empirical	0.000118	0.000244	-0.1399	5.5697
	estimated	0.000118	0.000244	-0.1399	5.5702
FTSE100	empirical	-0.000021	0.000145	-0.0450	5.9987
	estimated	-0.000021	0.000145	-0.0450	5.9993
CAC40	empirical	0.000103	0.000207	-0.0636	5.9566
	estimated	0.000103	0.000207	-0.0636	5.9560
IBEX35	empirical	0.000144	0.000202	-0.1350	6.0346
	estimated	0.000144	0.000202	-0.1350	6.0335
PX50	empirical	0.000335	0.000178	-0.0546	7.0230
	estimated	0.000335	0.000178	-0.0546	7.0237
BUX	empirical	0.000303	0.000286	-0.4920	11.8510
	estimated	0.000304	0.000286	-0.4921	11.8507
BETC	empirical	0.000771	0.000192	-0.3671	7.6766
	estimated	0.000771	0.000192	-0.3671	7.6779

One may notice that the estimated values of the main statistical indicators used in the financial theory to assess the return distribution (i.e., mean, variance, skewness, kurtosis) are indistinguishable from the empirical values (i.e., the values of the indicators calculated from the data).

Figure 1a shows the empirical probability distribution function (computed using non-parametric kernel methods), the one of the fitted Gaussian distribution and the one of the estimated Generalized Hyperbolic Distribution in case of BETC returns.

Figure 1. *The distribution of BETC returns*

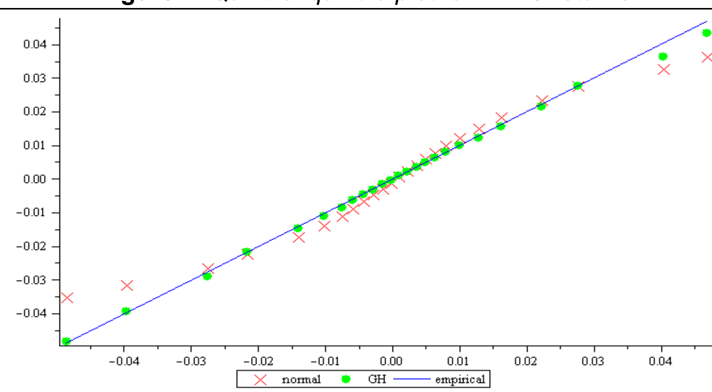


a. Probability distribution function | **b.** Log-probability distribution function

To better assess the tail behavior of BETC returns, figure 1b shows the logarithm of probability distribution functions. The tails of the Gaussian distribution decrease exponentially, but the tails of the estimated Generalized Hyperbolic Distribution evolve according to the power law of the empirical tails.

Figure 2 depicts the quintile-quintile plot for the Gaussian distribution and for the estimated Generalized Hyperbolic Distribution in the case of BETC returns.

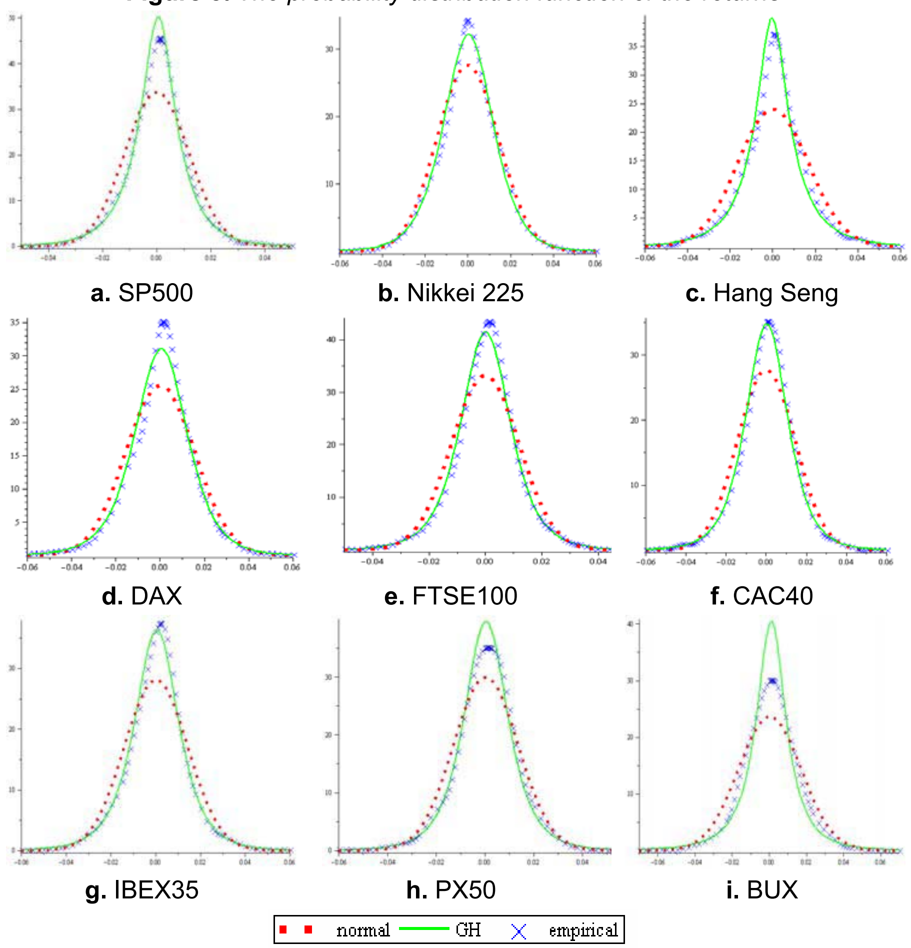
Figure 2. *Quintile–quintile plot for BETC returns*



The Generalized Hyperbolic Distribution represents a good approximation for the left tail of the empirical distribution, especially of the quintiles usually employed for VaR estimations (0.5%, 1%, 2.5%, and 5%).

Figure 3 presents the three probability distribution functions for the other nine stock index returns employed in this study.

Figure 3. *The probability distribution function of the returns*



One may notice in Figure 4 that the tails of the estimated Generalized Hyperbolic Distribution are a very good approximation for the evolution law of the empirical tails, both for index returns in developed countries and emerging economies.

Figure 4. *The log-probability distribution function of the returns*

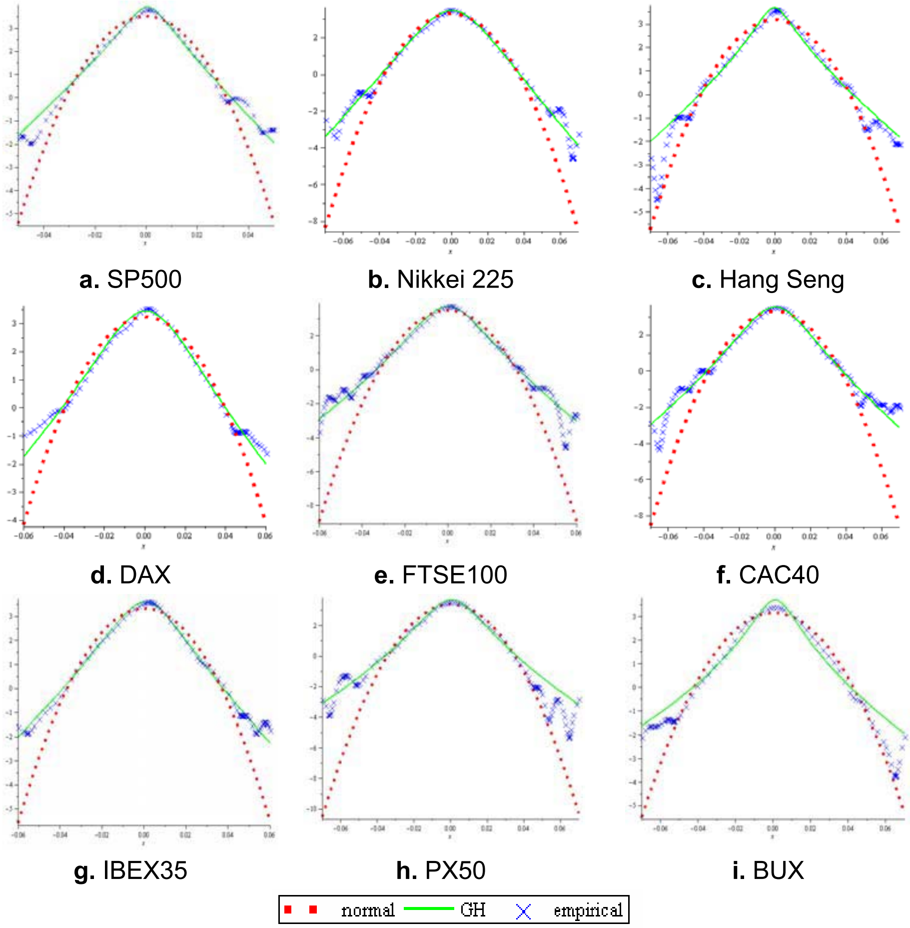


Table 4 presents the goodness-of-fit measures for the estimated Gaussian law and the estimated Generalized Hyperbolic Distribution. We employed two distance measures, the Kolmogorov-Smirnov distance and the Anderson-Darling distance.

Table 4

Goodness-of-fit measures

Index	KS distance		AD distance	
	normal	GH	normal	GH
SP500	0.0541	0.0183	4.4626	1.0154
Nikkei 225	0.0417	0.0191	2.7928	0.9059
Hang Seng	0.0691	0.0254	∞	1.6787
DAX	0.0617	0.0335	4.5234	1.7151
FTSE100	0.0563	0.0206	∞	1.2606
CAC40	0.0551	0.0186	4.3339	1.0689
IBEX35	0.0571	0.0226	4.5400	1.1457
PX50	0.0502	0.0259	∞	1.3274
BUX	0.0544	0.0361	∞	2.0417
BETC	0.0752	0.0179	4.9500	0.9546

As one may notice, irrespective of the measurement employed, the distance between the estimated Generalized Hyperbolic Distribution and the empirical distribution is lower than the distance between the normal distribution and the empirical one.

5. Concluding remarks

In the present study we estimated the parameters of the Generalized Hyperbolic Distribution for a series of stock index returns, including the Romanian BETC, and we investigated the goodness-of-fit of the estimated Generalized Hyperbolic Distribution to the empirical distribution. The empirical probability distribution function was estimated using non-parametric econometric methods.

The main finding is that, in comparison with the normal distribution, the Generalized Hyperbolic Distribution is a far better approximation of the empirical distribution. We obtained that the values of the main statistical indicators used in the financial theory to assess the return distribution (i.e., mean, variance, skewness, kurtosis) computed by means of the estimated Generalized Hyperbolic Distribution are indistinguishable from the empirical values (i.e. the values of the indicators calculated from the data). Consequently, the probability density function of the estimated Generalized Hyperbolic Distribution represents an almost exact approximation (at least up to the 4^{th} order term) of the empirical probability distribution function.

Also, according to the Kolmogorov-Smirnov and Anderson-Darling goodness-of-fit tests, the Generalized Hyperbolic Distribution is more appropriate to model the distribution of financial assets returns than the normal distribution.

The tail behavior of the Generalized Hyperbolic Distribution implies that the hyperbolic market model may be successfully employed to improve financial derivatives pricing models and the estimation of market risk using VaR methodology.

In order to quantify the market risk one has to take into consideration both the distribution of individual returns in the portfolio and the dependency between the assets. Consequently, further research will focus on modeling the dependency structure using copula functions.

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Appendix

The Kernel Density Estimation Methodology

In comparison with parametric estimators where the estimator has a fixed functional structure and the parameters of this function are the only information one needs to know, non-parametric estimators have no fixed structure and depend upon all the data points to reach an estimate. The histogram is the simplest and the most frequently encountered non-parametric density estimator. However, the histogram is not smooth and depends on the end points and the width of the bins. One can alleviate these problems by using kernel density estimators.

Kernel estimators smooth out the contribution of each observed data point over a local neighborhood of that point. The contribution of data point x_i to the density estimate at some point x depends on how far apart x_i and x are. The extent of this contribution is depending upon the shape of the kernel function adopted and the bandwidth. For a kernel function K and a bandwidth h, the estimated kernel density at any point x is given by (Rosenblatt, 1956; Parzen, 1962):

$$f_h(x) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{x - x_i}{h}\right) ,$$

where: the kernel function is symmetric and $\int_{-\infty}^{\infty} K(x)dx = 1$.

Although nowadays non-parametric kernel density estimation is a standard technique in econometrics, there is still a big dispute on how to asses the quality of the estimate and which is the optimal choice of the bandwidth. The quality of a kernel estimate depends less on the shape of K than on the value of its bandwidth h (Silverman, 1986). Small h values lead to very spiky estimates, while larger values lead to over-smoothing. In the present paper, we employed the Gaussian kernel with a bandwidth chosen according to the following "rule of thumb" (Silverman, 1986)

$$h = 1.06 \cdot \min\left(\sigma, \frac{R}{1.34}\right) \cdot n^{-1/5} ,$$

where: n is the sample size, σ^2 is the sample variance and R is the interquartile range (i.e. the difference between the third and first quartiles).