4. COST MINIMIZATION UNDER VARIABLE INPUT PRICES: A THEORETICAL APPROACH

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Abstract

It is generally admitted that fixed, low input prices for resources cause distortions in the input mix, in the sense of inefficient usage of resources.We consider a particular homogeneous functional form for representing the potential distortions in the input factor quantities in the context of deriving Cobb-Douglas cost functions and such a representation can offer a justification for why the average cost may behave eratically, altghough the technology remains unchanged. Fixed input prices become a special case. Our generalized form of Shepard[3]'s lemma allows us to interpret the corresponding input prices's homogeneity orders as measures of the efficiency wages.

Keywords: average cost, returns to scale, production,cost minimization, input supply function, marginal rate of technical substitution

JEL Classification: D24, C00, O30, O33

1.Introduction

In perfectly competitive markets, pursuing profit-maximization for a single-output firm whose production function exhibits increasing returns to scale is commonly regarded as being a necessary condition for achieving decreasing average cost. Also, the presence of increasing average cost is usually attributed to decreasing returns to scale production functions and it is connected with a long time horizon. Two basic assumptions are usually made (either explicitly or implicitly) in many microeconomics textbooks: constant factor prices and having a production function that meets the necessary conditions to allow the use of the Lagrange multiplier method for the constrained optimization problem: minimizing cost assuming technical efficiency. Most

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[3] *Altghough alternatively Shephard Lemma is used, we go with Shepard lemma, according to Mass-Collel (1995), pg.141.*

textbooks assume constant input prices either explicitly or implicitly. Input prices may be fixed for a number of reasons. The most common assumption is perfect competition in input markets.[4] Another assumption is administrative prices, set by a bureaucracy. As long as input prices remain fixed, the adjustment to a change in the demand for the inputs must be a pure quantity adjustment. If a firm's technology exhibits increasing returns to scale, then a decrease in factor prices will induce the firm to purchase greater input quantities, and output will increase. By increasing the scale of the operation, the firm can further reduce its average cost simply by moving right on the long-run average cost curve. This happens with fixed technology. Research by Erdeml (2009) explores this issue from a different perspective, concluding that increasing returns to scale are simply impossible. Shmanske (2012) also reaches this conclusion. Under the usual description of returns to scale, Erdeml and Shmanske are correct. The assumption used for returns to scale is usually stated as “Suppose all input quantities are exactly doubled. How much will output increase? ” Briefly consider the meaning of “all input quantities”. If we include raw materials, parts, intermediate goods and other physical objects it becomes clear that increasing returns to scale violates the law of conservation of mass:

“This law asserts that the mass…of a material particle remains invariant“(MIT, 2012).

If *all* input quantities are *exactly* doubled, the best we can ever do is to exactly double output. Thus we expect constant returns to scale most of the time. Inefficiencies can lead to decreasing returns to scale. But increasing returns to scale are impossible. We propose a slight variation that brings increasing returns back into economic theory. Instead of doubling the quantities of all inputs, why not double the quantities of capital and labor? That allows sufficient room to accommodate the law of conservation of mass.

For example, consider the wine industry. Increasing returns to scale are common in this industry. The reason is simple. The fundamental unit of capital in winemaking is the tank. For convenience, think of this as a cylinder 20 feet tall and 6 feet in diameter. The capital input is the surface area, 433.54 square feet. Output is proportional to volume, 565.49 cubic feet. Doubling the capital input implies increasing the surface area to 867.08. One way of achieving that surface area is to increase the diameter to 8 feet and length to 30.5 feet. With those dimensions, the volume of the cylinder will be 1,533.10, an increase by 2.71 times the original volume. As long as we get enough grapes to fill the cylinder, we will see increasing returns to scale (University of Huston, 2012).

We begin our analysis using a two-variable Cobb-Douglas production function. It is fairly straightforward to show that average cost can increase, decrease or remain constant. Textbooks generally demonstrate this by keeping input prices constant and using constant, increasing, and decreasing returns to scale. If the price of one unit of K ( ) or L ( ) is kept fixed, then all adjustments in the capital/labor markets will be

[4] *Even this assumption is somewhat puzzling. A seller in a perfectly competitive market faces a horizontal demand curve. However, the assumption made here is a horizontal input supply function. This is a less-explored aspect of perfect competition that bears further study.*

quantity adjustments. We show that allowing input prices to change can also create increasing, decreasing, or constant long-run average costs *even though returns to scale are constant.* This can also occur in the short run and may be misinterpreted as changes in technology. We next show that assuming homogeneous input prices is equivalent to assuming variable elasticity of the production function with respect to the input factors, K and L. Consequently, if the elasticity of the production function with respect to the input factors is not constant, then an alternative to considering production functions with variable elasticity is to assume fixed input factors elasticity of substitution with homogeneous input prices. That leads us to a generalization of the duality between the production function and the cost function under fixed input prices. The question addressed at the end of the section II is how one can distinguish between changes in technology and pressures in the input prices - when actually fixed input prices are displayed?

The assumption of horizontal input supply curves, corresponding to a situation of perfect competition on the supply side, is demonstrably not true. The simplest example is the method many firms use to increase the quantity of labor: asking workers to work more than eight hours per day and/or 40 hours per week. This request is usually accompanied by an offer of a higher wage rate for the overtime worked. In the U.S. it's the law:

"An employer who requires or permits an employee to work overtime is generally required to pay the employee premium pay for such overtime work. Employees covered by the Fair Labor Standards Act (FLSA) must receive overtime pay for hours worked in excess of 40 in a workweek of at least one and one-half times their regular rates of pay. The FLSA does not require overtime pay for work on Saturdays, Sundays, holidays, or regular days of rest, unless overtime hours are worked on such days.

The FLSA, with some exceptions, requires bonus payments to be included as part of an employee's regular rate of pay in computing overtime.

Extra pay for working weekends or nights is a matter of agreement between the employer and the employee (or the employee's representative). The FLSA does not require extra pay for weekend or night work or double time pay."[5]

Thus, in many industries and even in the short run, increasing the quantity of labor often causes an increase in the wage. Whether this is caused by supply and demand or legislation is largely irrelevant to our analysis.This is not the case for Romania or other emerging or developing countries. The increase in wage, yet, is not done instantaneously - but after pressures excised by the workers, for example. Although it is acknowledged that overtime work might conduce to inefficiencies, these are not comprised in an analytical model. This paper offers a formalization which could also respond to this issue.

In section II we illustrate the derivation of the cost function assuming variable input prices and a two-variable Cobb-Douglas production function. The resulting cost function is then generalized for separable, homogeneous production functions of an

[5] *http://www.dol.gov/dol/topic/wages/overtimepay.htm. [Accessed in June 2012].*

arbitrary number of variables and, later, for homothetic production functions. While deriving these results, we also expand on the conditional demand functions for the inputs, Shephard's lemma, and the well-known result that, in equilibrium, the marginal rate of technical substitution equals the ratio of the corresponding input prices.

In section III we derive sufficient conditions for the construction of the cost function under a particular quasi-convex production function as an illustration of the fact that under fixed input prices there is still possible for "un-orthodox" methods of producing to yield well-behaved average cost curves.

Conclusions point to eventual econometric applications in the context of the relevant findings for the Romanian economy.

2. The Cost Function Assuming Homogeneous Input Prices

In section II.1 we derive a cost function using a simple two-input Cobb-Douglas production function. Section II.2 generalizes these results to an arbitrary number of inputs. We conclude with a discussion of the important role of homothetic production functions in II.3.

II.1 The Cost Function for a Two-Variable Cobb-Douglas Production Function

Assume a two variable Cobb-Douglas production function

$$f(K,L) = AK^{\alpha}L^{\beta} \quad (2.1.1)$$

For reasons that will soon become clear, we call this the *unenhanced Cobb-Douglas production* function. This production function is linearly separable, homogeneous of order r = $\alpha + \beta$ and concave iff $0 < \alpha, \beta < 1$ and

Assume that the input prices for capital and labor depend on the quantities of capital and labor, respectively, as in the following notation:

$$w_K = w_K(K), w_L = w_L(L) \quad (2.1.2)$$

Such an assumption could be made, for instance, in situations in which input administered prices operate and there are discrepancies with respect to the real, market input prices. In this circumstance, either explicit or implicit correspondent quantity adjustment might occur.

If these price functions are both assumed to be homogeneous of order and then

$$w_K = w_K(K) = w_K(K \cdot 1) = K^{h_1}w_K(1) \quad \text{and}$$
$$w_L = w_L(L) = w_L(L \cdot 1) = L^{h_2}w_L(1) \quad (2.1.3)$$

In the previous formulation, w_K can be fathomed with the sticky input price (eventually, some administered price) and $w_K(1)$ is the real, market price. The same applies for w_L and $w_L(1)$.

In this context, $w_K(1)$ is the price paid for one unit of capital, K, and for brevity we denote this as $\overline{w_K}$. Similarly, $w_L(1)$ is the real price paid for one unit of labor, L, and we denote this as $\overline{w_L}$.

If $\overline{w_L} < w_L$ then in the previous context $h_2 > 0$ and this can be read as follows: if the real market input price for labor, $\overline{w_L}$ is less than the fixed, administered input price w_L then the firm might be able to have more input factor L, either explicitely or implicitely-for example, in this case-through informally imposing longer hours.

If, on the other hand the real market input price is greater than the fixed, administered one, $\overline{w_L}>w_L$ then $h_2 < 0$. This could happen for instance when labor force is underpaid (in the budgetary sectors, for eg.) and decides to leave elsewhere, or simply work effectively less hours in the due time.If $h_2 = 0$ the sticky input price w_L is just the real, market price $\overline{w_L}$.

The point elasticity of input prices with respect the corresponding input factors are:

$$\varepsilon_{w_K:K} = \frac{\partial w_K / \partial K}{w_K / K} \text{ and } \varepsilon_{w_L:L} = \frac{\partial w_L / \partial L}{w_L / L} \tag{2.1.4}$$

Clearly $\varepsilon_{w_K:K} = h_1$ if and only if there is some constant *M* so that $w_K = MK^{h_1}$. If $M = w_K(1)$ then then this corresponds to the assumption that the price of capital is homogeneous of order $\mathbf{h_1}$. The order of homogeneity can be interpreted as the point elasticity of the input price with respect to the corresponding input factor, which will be constant. In particular, input prices are constant if and only if the order of homogeneity is zero or, equivalently, the point elasticity is zero.

Assuming the production function is also quasi-concave (or α+β<1), the cost function will be the solution to the minimization problem

$$\min(w_K(K) \cdot K + w_L(L) \cdot L) \text{ subject to } f(K, L) = q \tag{2.1.5}$$

In the Cobb-Douglas case, the total cost function is:

$$C(q, w_K(1), w_L(1)) = B \cdot q^{\frac{1}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)}} \cdot w_K(1)^{\frac{\frac{\alpha}{h_1+1}}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)}} \cdot w_L(1)^{\frac{\frac{\beta}{h_2+1}}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)}} \tag{2.1.6}$$

with

$$B = \left(\frac{1}{A}\right)^{\frac{1}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)}} \left[\left(\frac{\alpha}{\beta} \frac{h_2+1}{h_1+1}\right)^{\frac{\frac{\beta}{h_2+1}}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)}} + \left(\frac{\beta}{\alpha} \frac{h_1+1}{h_2+1}\right)^{\frac{\frac{\alpha}{h_1+1}}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)}} \right] \tag{2.1.7}$$

It is now clear that the cost function in (2.1.6) corresponds to the cost function associated with the fixed input prices $\overline{w_L} = w_L(1)$, $\overline{w_K} = w_K(1)$. This leads to the *enhanced Cobb-Douglas production function*:

$$f(K,L) = AK^{\frac{\alpha}{h_1+1}}L^{\frac{\beta}{h_2+1}} \tag{2.1.8}$$

Average cost is

$$AC = \frac{C}{q} = B \cdot q^{\frac{1}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)}-1} g(w_K(1), w_L(1)) \tag{2.1.9}$$

with

$$g(w_K(1), w_L(1)) = w_K(1)^{\frac{\frac{\alpha}{h_1+1}}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)}} \cdot w_L(1)^{\frac{\frac{\beta}{h_2+1}}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)}} \tag{2.1.10}$$

and the sign of the average cost curve's slope is therefore the sign of the exponent

$$\frac{1}{\left(\frac{\alpha}{h_1+1}\right)+\left(\frac{\beta}{h_2+1}\right)} - 1 \text{ in} \quad (2.1.7).$$

For example, suppose the unenhanced production function exhibits increasing returns to scale ($\alpha = 0.5$, $\beta = 0.75$). If $h_1 = 0.125, h_2 = 0.25$ then the enhanced production function will exhibit economies of scale (decreasing average costs). On the other hand, if $h_1 = 0.5$, $h_2 = 1.1$ diseconomies of scale (increasing average costs) occur. This happens because homogeneity in input prices now has an impact on the effective production function's capital and labor elasticity. So, if input factor prices are homogenous of order h_1 and h_2 the enhanced production function is $f(K,L) = AK^{\frac{\alpha}{h_1+1}}L^{\frac{\beta}{h_2+1}}$. If a researcher assumes $f(K,L) = AK^{\alpha}L^{\beta}$, the estimated production function will have incorrect parameters. And even though $\alpha + \beta > 1$, there will always be values of $h_1$and h_2 , that can cause the average cost function to slope upward, slope downward, or be horizontal.

Input prices may be "sticky" (Blinder, 1982, Mortenson, 1970. Phelps,1968). With sticky prices, the signal of changes in demand is a change in quantity of the input purchased. This may induce changes in average cost caused by either returns to scale or price pressures. Such average cost changes may be regarded as random when, in fact, they are signaling pressures on input prices. In order to test for homogeneity of order different from zero in the input prices, one can proceed, for instance, like this:

1. Estimate $\hat{\alpha}$, $\hat{\beta}$ from the production function
   $$\ln q = lnA + \alpha \ln K + \beta \ln L \tag{2.1.11}$$
2. Estimate the cost function
   $$\ln C = lnB + c_1 \ln q + c_2 \ln w_L + c_3 \ln w_K \tag{2.1.12}$$

3. Jointly test whether $c_1 = \frac{1}{\hat{\alpha}+\hat{\beta}}$, $c_2 = \frac{\hat{\beta}}{\hat{\alpha}+\hat{\beta}}$, $c_3 = \frac{\hat{\alpha}}{\hat{\alpha}+\hat{\beta}}$.

2.2. Generalization to a Cobb-Douglas Type Production Function of an Arbitrary Number of Variables

Extending the previous section to a production function[6] of m-variables of the form

$$f(z_1, z_2, \ldots z_m) = f_1(z_1) \cdot f_2(z_2) \cdot \ldots \cdot f_m(z_m) \tag{2.2.1}$$

where $f_i(z_i)$ is a homogeneous function of order α_i, for all $i = 1, \ldots m$. Such a production function is homogeneous of order r. with $= \alpha_1 + \ldots \alpha_m$ . .

So, assuming a production function of the form (2.2.1) and also assuming that the the input prices $(w_i(z_i))_{i=1,m}$ are all homogeneous functions of the correspondent orders $(h_i)_{i=1,\ldots,m}$,the cost function which is the solution to the minimization problem

$$\min(w_1(z_1) \cdot z_1 + \ldots + w_m(z_m) \cdot z_m) \text{ subject to } f(z_1, \ldots z_m) = q \tag{2.2.2}$$

is as follows:

$$C(q, w_1(1), \ldots w_m(1)) = \frac{1}{C} \cdot q^C \, w_1(1)^{C \cdot \frac{\alpha_1}{h_1+1}} \cdot \ldots \cdot w_m(1)^{C \cdot \frac{\alpha_m}{h_m+1}} \cdot D \tag{2.2.3}$$

with

$$C = \frac{1}{\sum_{i=1}^{m} \frac{\alpha_i}{h_i+1}} \tag{2.2.4}$$

and

$$D = \left[\frac{1}{\prod_{i=1}^{m} f_i(1)}\right]^C \prod_{i=1}^{m} \left(\frac{h_i+1}{\alpha_i}\right)^{C \cdot \frac{\alpha_i}{h_i+1}} \tag{2.2.5}$$

The corresponding conditional demand function for the optimal inputs z_i , z_i^{opt} is as follows:

$$z_i^{opt} = D^{\frac{1}{h_i+1}} \left(\frac{\alpha_i}{h_i+1}\right)^{\frac{1}{h_i+1}} q^{\frac{C}{h_i+1}} \cdot w_1(1)^{\frac{\alpha_1}{h_1+1} \cdot C \cdot \frac{1}{h_i+1}} \cdot \ldots \cdot w_m(1)^{\frac{\alpha_m}{h_m+1} \cdot C \cdot \frac{1}{h_i+1}} \tag{2.2.6}$$

for every i from 1 to m.

Once again, assume the input price of the production factor i, w_i- to be homogeneous of order h_i, so that $w_i(z_i) = z_i^{h_i} w_i(1)$. This implies that instead of spending w_i for one unit of z_i and receiving z_i, the firm is actually in receipt of $z_i^{h_i+1}$ at the market real input price $w_i(1)=\overline{w_i}$. The result in (2.2.7) is a generalization of Shephard's lemma (the rate of increase of cost with respect to input price equals the quantity of that input that the

[6] *All over this paper the term production function refer to a function defined on a production set with all the necessary conditions so that the cost function checks all the usual good properties-as for example Propositon 5.C.2 in Mas-Colell (1995), pg. 140.*

firm employs when it is at the cost-minimizing combination) supports this interpretation:

$$\frac{\partial C(q,w_1(1),..,w_m(1))}{\partial w_i(1)} = \left(z_i^{opt}\right)^{h_i+1} \tag{2.2.7}$$

Naturally, this result only holds for the specific production function considered above.

The standard equilibrium condition is that the marginal rate of technical substitution of input *j* with input *i* is equal to the ratio of the corresponding input prices as long as input prices are constant ($\overline{w_i} = w_i(1)$ and $\overline{w_j} = w_j(1)$) :

$$MRTS_{ij} = \frac{MP_i}{MP_j}\left(z_1^{opt},\dots,z_m^{opt}\right) = \frac{w_i(1)}{w_j(1)} \tag{2.2.8}$$

We generalize this result in the context of homogeneous input prices and the enhanced Cobb-Douglas production function:

$$MRTS_{ij} = \frac{MP_i}{MP_j}\left(z_1^{opt},\dots,z_m^{opt}\right) = \frac{w_i(1)^{\frac{1}{h_i+1}}}{w_j(1)^{\frac{1}{h_j+1}}} \cdot q^{C \cdot \frac{h_i-h_j}{(h_j+1)(h_i+1)}} \cdot E \tag{2.2.9}$$

with

$$E = D^{\cdot \frac{h_i-h_j}{(h_j+1)(h_i+1)}} \cdot \frac{(h_i+1)^{\frac{1}{h_i+1}}}{(h_j+1)^{\frac{1}{(h_j+1)}}} \cdot \frac{\alpha_i^{1-\frac{1}{h_i+1}}}{\alpha_j^{1-\frac{1}{h_j+1}}} \tag{2.2.10}$$

and the constants C and D as defined in (2.2.4) and (2.2.5).

Equation (2.2.10) shows that under the assumption of different homogeneity orders for the input-prices, the marginal rate of substitution is also dependent on the quantity produced. The fact that this result depends on our use of the Cobb-Douglas functional form is irrelevant. This alone constitutes sufficient proof for the result that under variable input prices at equilibrium the MRTS depends on the quantity produced as well as the real,market input prices.If neither the corresponding orders of homogeneity i and j nor the real input market prices $\overline{w_i}$, $\overline{w_i}$ are known then the equilibrium condition for sticky input prices does not hold true, altghough the firm might minimize its costs.

To summarize, we assumed a cost-minimizing firm producing a single output in a competitive market and facing a multivariable Cobb-Douglas production function. If the input prices are sticky while the value of the input factors expressed through their market input prices changes then the input supply function lead to real, market input prices of different orders of homogeneity.In such a case,even if the production function has increasing returns to scale, average cost may increase, decrease, or remain unchanged.

II.3 Homothetic Production Functions

If f(.) is a homothetic, concave production function, then under fixed input prices the cost function is separable into the level of production (*q*) and input prices. More precisely, “If $f(z_1,..,z_n)$ is a homothetic production function, then assuming profit-maximization with fixed input prices, the cost function $C(w_1,..,w_m,q) = a(q)b(w_1,..,w_m)$" (Cowell 2006,46).

This property allows one to determine in gneral the slope of the average cost function using the exhibited returns to scale and the horizontal input supply function.

In this section it is illustrated that if we extend the assumption of fixed input prices to homogeneous input prices of the same order the general property cited above still hold true.Following the same path of argument,if the concave production function is not only homothetic, but also separable, the result derived above can be extended to variable, homogeneous input prices, with different orders of homogeneity.

Consider the problem

$$\min(w_1(z_1)\cdot z_1 + \dots + w_m(z_m)\cdot z_m) \text{ subject to } f(z_1, \dots z_m) = q \tag{2.3.1}$$

Assume that the production function $f(z_1, \dots, z_m)$ is quasi-concave, homothetic of order r and the input prices as functions of the production factors $w_1(z_1), \dots, w_m(z_m)$ are homogeneous of order h such that ($w_i(z_i) = z_i^h w_i(1)$, $w_i(1) = w_i$)

We next derive the implicit form of the cost function assuming that the constrained minimization problem above has an economic meaningful solution. The first-order conditions provide a unique solution. The second order conditions will be met as long as *h* > 0 (the function to be minimized is quasi convex if *h* > 0).

If $w_i(z_i)$ is homogeneous of order h then $w_i(z_i) = z_i^h w_i(1)$ and f'_{z_i} is a homogeneous function of order $r - 1$, for every every i = 1, …, m.

The Lagrangean function $L(z_1, \dots, z_m, \lambda)$ is as usual

$$L(z_1, \dots z_m, \lambda) = w_1(z_1)\cdot z_1 + \dots + w_m(z_m)\cdot z_m + \lambda\cdot(q - f(z_1, \dots z_m))$$
$$= z_1^{h+1} w_1(1) + \cdots z_1^{h+1} w_1(1) + \lambda\cdot(q - f(z_1, \dots z_m))$$

The partial derivatives with respect to all its arguments are as follows

$$L'_{z_1}(z_1, \dots z_m, \lambda) = (h+1) z_1^h w_1(1) - \lambda\cdot f'_{z_1}(z_1, \dots z_m) = (h+1) z_1^h w_1(1) - \lambda\cdot z_1^{r-1} f'_{z_1}\left(1, \frac{z_2}{z_1}, \dots \frac{z_m}{z_1}\right) \tag{2.3.2.1}$$

$$L'_{z_i}(z_1, \dots z_m, \lambda) = (h+1) z_i^h w_i(1) - \lambda\cdot f'_{z_i}(z_1, \dots z_m) = (h+1) z_i^h w_i(1) - \lambda\cdot z_1^{r-1} f'_{z_i}\left(1, \frac{z_2}{z_1}, \dots \frac{z_m}{z_1}\right), \text{ for every } i = 2, \dots m \tag{2.3.2.i}$$

$$L'_{\lambda}(z_1, \dots z_m, \lambda) = q - f(z_1, \dots z_m) = q - z_1^r f(1, \frac{z_2}{z_1}, \dots \frac{z_m}{z_1}) \tag{2.3.2.λ}$$

By using (2.3.2.i) and (2.3.2.1) for every $i = 2, \dots, m$, the first order conditions

$$L'_{z_1}(z_1, \dots z_m, \lambda) = 0 \text{ and } L'_{z_i}(z_1, \dots z_m, \lambda) = 0 \qquad (2.3.2)$$

yield

$$\left(\frac{z_i}{z_1}\right)^h \frac{w_i(1)}{w_1(1)} = \frac{f'_{z_i}(1, \frac{z_2}{z_1}, \dots \frac{z_m}{z_1})}{f'_{z_1}(1, \frac{z_2}{z_1}, \dots \frac{z_m}{z_1})} \qquad (2.3.3)$$

Let

$$t_i = \frac{z_i}{z_1} \qquad (2.3.4)$$

then using (2.3.4), equation (2.3.3) becomes

$$(t_i)^h \cdot \frac{w_i(1)}{w_1(1)} = \frac{f'_{z_i}(1, t_2, \dots t_m)}{f'_{z_1}(1, t_2, \dots t_m)} \text{ for every } i = 2, \dots m \qquad (2.3.5)$$

and from (2.3.2.λ) and (2.3.5) the condition

$$L'_{\lambda}(z_1, \dots z_m, \lambda) = 0 \qquad (2.3.6)$$

becomes

$$q = z_1^r f(1, t_2, \dots t_m) \qquad (2.3.7)$$

By the theorem of implicit functions[7] (2.3.5) yields

$$t_i = g_i(w_i(1), w_1(1), h) \text{ for every } i = 2, \dots m \qquad (2.3.8)$$

where the function g_i depends of the functions f'_{z_i} and f'_{z_1}.

Substituting (2.3.8) into (2.3.7) we derive *the conditional demand for input* (demand that is conditional on *q*), as

$$z_1^{opt} = z_1^{opt}(q, w_1(1), \dots, w_m(1)) == q^{\frac{1}{r}}\left[\frac{1}{f(1, g_2(w_2(1), w_1(1), h), \dots g_m(w_m(1), w_1(1), h))}\right]^{\frac{1}{r}} \qquad (2.3.9)$$

The *conditional demand for input* z_i, z_i^{opt} can be derived by from replacing (2.3.8) and (2.3.9) into (2.3.4)

$$z_i^{opt} = z_i^{opt}(q, w_1(1), \dots, w_m(1)) =$$
$$= q^{\frac{1}{r}} g_i(w_i(1), w_1(1), h)\left[\frac{1}{f(1, g_2(w_2(1), w_1(1), h), \dots, g_m(w_m(1), w_1(1), h))}\right]^{\frac{1}{r}} \qquad (2.3.10)$$

The cost function will then be

[7] *Once again, here has to be mentioned that if the production function check the previous cited conditions in Mas-Colell (pg 140) then the theorem of implicit functions cand be properly applied. See also a similar justification in Cowell,F. (2005),pg. 507.*

$$C\left(q,w_1(1),\dots w_m(1)\right) = w_1\left(z_1^{opt}\right)\cdot z_1^{opt} + \dots + w_m\left(z_m^{opt}\right)\cdot z_m^{opt} =$$
$$= (z_1^{opt})^{h+1}w_1(1) + (z_2^{opt})^{h+1}w_2(1) + \dots + (z_m^{opt})^{h+1}w_m(1)$$
$$= q^{\frac{h+1}{r}}B(w_1(1),\dots,w_m(1)) \qquad (2.3.11)$$

with

$$B\left(w_1(1),\dots,w_m(1)\right)$$
$$= [\frac{1}{f\left(1,g_2(w_2(1),w_1(1),h),\dots,g_m(w_m(1),w_1(1),h)\right)}]^{\frac{1}{r}}\left[w_1(1) + \sum_{i=2}^{n} w_i(1)\cdot g_i(w_i(1),w_1(1),h)\right] \qquad (2.3.12)$$

Then the average cost is

$$AC = \frac{C(q,w_1(1),\dots w_m(1))}{q} = q^{\frac{h+1}{r}-1}B(w_1(1),\dots,w_m(1)) \qquad (2.3.\ 13)$$

and the slope for the average cost is

$$(AC)'_q = \frac{C(q,w_1(1),\dots w_m(1))}{q} = (\frac{h+1}{r} - 1)q^{\frac{h+1}{r}-2}B(w_1(1),\dots,w_m(1)) \qquad (2.3.14)$$

It is beyond the scope of our current work to determine whether these specific properties can be generalized without using a specific functional form for the production function. In the following sections we consider some uncommon specification. Despite this, we can at least infer some facts about the link between a cost function and the quantity of output. Specifically, we will examine the relationship between returns to scale and cost.

From (2.3.11) we see that the cost function is increasing with q since we assumed $h > 0$.

Also, if the production function has constant returns to scale ($r = 1$) then average cost is independent of q if and only if the input prices are constant.

If the production function has constant returns to scale ($r = 1$) and input prices are homogeneous of order $h > 0$ then $(AC)'_q > 0$ and the firm exhibits diseconomies of scale (as output increases, average cost increases).The condition $h > 0$ means that instead of paying w_i for one unit of the input factor z_i and getting z_i units of that input, the firm gets z_i^h effective units of input i. If z_i is greater than one, we can generalize the idea of efficiency wages to efficiency input prices. Efficiency input prices mean that, for a given expenditure on an input i, a larger effective quantity of input i is obtained. This result will act in the opposite direction if $z_i < 1$.

If the production function exhibits increasing returns to scale (homogeneous of order $r > 1$ or homothetic of order $r > 1$) then the firm will exhibit economies of scale if and only if ($h + 1 - r/r < 0$) which necessarily implies $h < r - 1$.

In practice, we find it difficult to believe that any production function has precisely constant returns to scale. If it has slightly increasing to scale (say r - 1 = ε, a very small number) and input prices are slightly efficient ($h < r - 1$) economies of scale occur. But if wages are overly efficient (say $h - (r - 1) =$ ε, a very small number) then diseconomies of scale occur. Similar results obtain for $r < 1$ and close to 1.

In particular if input prices are constant ($h = 0$) or homogeneous of order $h < 0$, the condition $h < r - 1$ is automatically fulfilled.

3. Sufficient Conditions for Deriving Cost Functions For a Quasi-Convex Production Function

In the following it is illustrated how, for a particular example of quasi-convex production function[8] it can be derived the correspondent cost function with its usual good proprieties, under the generalized hypothesis of homogeneous input prices.

Consider the production function

$$f(z_1, z_2) = z_1^2\left(1 + \frac{1}{z_2}\right) \tag{3.1}$$

which has increasing returns to scale[9], is continuous, differentiable and quasi-convex[10]. Therefore,under the assumption of fixed input prices it is not possible to derive a solution to the cost-minimization problem by appliying the lagrange multiplier method. We now proceed to a solution using the usual Lagrange multiplier technique under the assumption of the homogeneous input prices.

For the production function in (3.1) the problem of minimizing is

$$\min(w_1(z_1)\cdot z_1 + w_2(z_2)\cdot z_2) \text{ subject to } f(z_1, z_2) = q \text{ and } z_1, z_2 \geq 0 \tag{3.2}$$

It will be kept the assumption of input-prices homogeneous of the h_1, h_2 correspondent orders:

$$w_1(z_1) = z_1^{h_1} w_1(1) \text{ and } w_2(z_2) = z_2^{h_2} w_2(1) \tag{3.3}$$

The Lagrangean function corresponding to the constrained optimization problem in (3. 2) is

$$L(z_1, z_2, \lambda) = z_1^{h_1+1} w_1(1) + z_2^{h_2+1} w_2(2) + \lambda(q - f(z_1, z_2)) \tag{3.4}$$

The first order conditions are as follows:

$$L'_{z_1} = 0, \quad L'_{z_2} = 0, \quad L'_{\lambda} = 0 \tag{3.5}$$

[8] *Apart of the previously mentioned conditions assumed for production function ,the quasi-convex /quasi concave production function mentioned above follow the definitions in Cowell, F. (2005), pp. 504-504.*

[9] *The proof is available in Appendix 1, section 4.*

[10] *The proof is available in Appendix 1, section 4.*

with

$$L'_{z_1} = (h_1 + 1) z_1^{h_1} w_1(1) - \lambda f'_{z_1} \qquad (3.6)$$

$$L'_{z_2} = (h_2 + 1) z_2^{h_2} w_2(1) - \lambda f'_{z_2} \qquad (3.7)$$

$$L'_{\lambda} = q - f(z_1, z_2) \qquad (3.8)$$

From the system (3.6)-(3. 8) one gets that if there is some optimal (z_1^{opt}, z_2^{opt}) solution to the constrained optimization problem (3.2), then the two components are also checking the equation

$$\frac{z_1^{h_1+3}}{z_2^{h_2+2}} = -2\frac{h_2+1}{h_1+1} q w_1(1)^{-1} w_2(1) \qquad (3.9)$$

One could get z_2 on z_1 from (3.9) but then replacing the previously determined general dependence in the equation (3.8) leaves not too much room for finding an analytical solution. Therefore, the question we raise is the next one: *is it available some convenient combination of values for h_1 and h_2 so that the cost function derived as a solution to the constrained minimization problem in (3.2) has the "good" properties usually available when one is assuming fixed input prices and quasi-concave production function?* It shall be proved that the answer is positive and this is interpreted as: under some favourable market conditions it is possible that abnormal methods of producing yield well-behaved average cost-curves.

In the following we show that it is possible to find such a combination of h_1 and h_2 (the order of homogeneity for the input prices) so that the corresponding cost function is well behaved.

In (3.9) assume $h_2 = -2$. Then, the unique solution of the (3.6)-(3.8) is

$$z_1^{opt} = \left(\frac{2}{h_1+1}\right)^{\frac{1}{h_1+3}} q^{\frac{1}{h_1+3}} w_1(1)^{-\frac{1}{h_1+3}} w_2(1)^{\frac{1}{h_1+3}} \qquad (3.10)$$

$$z_2^{opt} = \frac{(z_1^{opt})^2}{q-(z_1^{opt})^2} \qquad \left(0 \le z_1^{opt} \le \sqrt{q}\right) \qquad (3.11)$$

$$\lambda^{opt} = \frac{w_2(1)}{(z_1^{opt})^2} \qquad (3.12)$$

The cost function is

$$C(w_1(1), w_2(1), q) = w_1(z_1^{opt}) \cdot z_1^{opt} + w_2(z_2^{opt}) \cdot z_2^{opt} \qquad (3.13)$$

$$= q^{\frac{h_1+1}{h_1+3}} w_1(1)^{\frac{2}{h_1+3}} \left[\left(\frac{2}{h_1+1}\right)^{\frac{h_1+1}{h_1+3}} w_2(1)^{\frac{1}{h_1+3}} + \left(\frac{2}{h_1+1}\right)^{-\frac{2}{h_1+3}} w_2(1)^{\frac{h_1+1}{h_1+3}}\right] - w_2(1) \qquad (3.14)$$

The average cost is

$$AC = AC(w_1(1), w_2(1), q) = \frac{C(w_1(1), w_2(1), q)}{q} =$$

$$q^{-\frac{2}{h_1+3}} w_1(1)^{\frac{2}{h_1+3}} \left[\left(\frac{2}{h_1+1}\right)^{\frac{h_1+1}{h_1+3}} w_2(1)^{\frac{1}{h_1+3}} + \left(\frac{2}{h_1+1}\right)^{-\frac{2}{h_1+3}} w_2(1)^{\frac{h_1+1}{h_1+3}} \right] - \frac{1}{q} w_2(1) \qquad (3.15)$$

and the derivative of the average cost with respect of the quantity produced, q , is

$$(AC)'_q = -\frac{2}{h_1+3} q^{-\frac{h_1+5}{h_1+3}} w_1(1)^{\frac{2}{h_1+3}} \left[\left(\frac{2}{h_1+1}\right)^{\frac{h_1+1}{h_1+3}} w_2(1)^{\frac{1}{h_1+3}} + \left(\frac{2}{h_1+1}\right)^{-\frac{2}{h_1+3}} w_2(1)^{\frac{h_1+1}{h_1+3}} \right] +$$
$$+\frac{1}{q^2} w_2(1) \qquad (3.16)$$

The derivative of the cost function with respect to $w_1(1)$ is

$$\frac{\partial C}{\partial w_1(1)} = \left(\frac{2}{h_1+1}\right)^{\frac{h_1+1}{h_1+3}} q^{\frac{h_1+1}{h_1+3}} w_1(1)^{-\frac{h_1+1}{h_1+3}} w_2(1)^{\frac{1}{h_1+3}} \frac{2}{h_1+3} \left[1 + \frac{h_1+1}{2} w_2(1)^{\frac{h_1}{h_1+3}}\right] \qquad (3.17)$$

If one would want to check whether the Shepard Lemma, in its generalized form holds true for the particular production function in (3.1) :

$$\frac{\partial C}{\partial w_1(1)} = (z_1^{opt})^{h_1+1} \qquad (3.18)$$

Since from (3.10),

$$(z_1^{opt})^{h_1+1} = \left(\frac{2}{h_1+1}\right)^{\frac{h_1+1}{h_1+3}} q^{\frac{h_1+1}{h_1+3}} w_1(1)^{-\frac{h_1+1}{h_1+3}} w_2(1)^{\frac{h_1+1}{h_1+3}} \qquad (3.19)$$

then for (3.18) to hold true, one should search for h_1 so that

$$\frac{2}{h_1+3}\left[1 + \frac{h_1+1}{2} w_2(1)^{\frac{h_1}{h_1+3}}\right] = 1 \qquad (3.20)$$

It can be easily observed that (3.20) holds true for every value of $w_2(1)$ if $h_1 = 0$.

Therefore, for $h_1 = 0$ and $h_2 = -2$ the unique solution to the system (3.6)-(3.8) is

$$z_1^{opt} = \sqrt[3]{2}\, q^{\frac{1}{3}} w_1(1)^{-\frac{1}{3}} w_2(1)^{\frac{1}{3}} \qquad (3.21)$$

$$z_2^{opt} = \frac{1}{\frac{1}{\sqrt[3]{4}} q^{\frac{1}{3}} w_1(1)^{\frac{2}{3}} w_2(1)^{-\frac{2}{3}} - 1} \qquad (0 \le z_1^{opt} \le \sqrt{q}) \qquad (3.22)$$

$$\lambda^{opt} = \frac{1}{\sqrt[3]{4}} q^{-\frac{4}{3}} w_1(1)^{\frac{4}{3}} w_2(1)^{\frac{1}{3}} \qquad (3.23)$$

The corresponding cost function as in (3.13) is, for the particular situation above,

$$C(w_1(1), w_2(1), q) = q^{\frac{1}{3}} w_1(1)^{\frac{4}{3}} w_2(1)^{\frac{1}{3}} \left[2^{\frac{1}{3}} + 2^{-\frac{4}{3}}\right] - w_2(1) \qquad (3.24)$$

and the correspondent generalizations for the Shepard Lemma hold true in this particular situation for z_1^{opt} and z_2^{opt} in (3.21) and (3.22):

$$\frac{\partial C}{\partial w_1(1)} = z_1^{opt}, \quad \frac{\partial C}{\partial w_2(1)} = (z_2^{opt})^{-1} \qquad (3.25)$$

Also, the well-known interpretation associated with the optimal value of the Lagrange multiplier holds true in this particular context:

$$\frac{\partial c}{\partial q} = \lambda^{opt} \tag{3.26}$$

In other words, given some “un-orthodox” methods of producing, there is still the possibility that, under sticky input prices, average cost will look the “normal way”. This challenges the duality relationship between the production function and average cost.

IV. Conclusions

In this paper we demonstrated conclusively that economic theory might not be consistent with the one-to-one correspondence between returns to scale and the slope of the average cost function under sticky input prices. We developed an enhanced production function that incorporates parameters of homogeneous input supply functions. We have shown that, if supply functions are homogeneous of some other order than zero, the well known result that MRTS equals the ratio of the input prices, under profit maximization and concave production function, no longer holds. Instead, MRTS depends on both input prices and quantities. Finally, on a particular convex production function, potentially assimilated with non-orthodox methods of production, we derived sufficient conditions for the existence of a cost function.

These findings might be empirically tested in the case of Romanian economy, where few important sectors experiences input-factor administered prices (e.g. energy sector). since some of the most influential econometric studies for this country (Dobrescu,2006, Dobrescu, 2009) rely on the Cobb-Douglas specification for production functions. Another potential development of the theory presented in this paper would be the direction of studing the stability of the I-O coefficients under the presence of administered prices. There are two important passages in Dobrescu which point in this direction:

“The temporal behaviour of I-O coefficients is yet an open question. In most applications, the stability of matrix A is usually assumed. This comes from both classical and extended interpretations of the Cobb-douglas production function. ”

and

“Like other previous studies, the analysis of Romanian I-o tables confirms that the technical coefficients are volatile. What needs to be documented is the nature of volatility and the highly questionable factor is the presence of non-linearities in the respective statistical series.”

Indeed, the effect of administered prices, according to the results presented in our paper is also non-linear in the output elasticities with respect to the input factors. It is author’s hope that further empirical studies using enhanced Cobb-Douglas production function instead of the traditional one may shed interesting directions of further theory development.

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Appendix 1

Derivation of the cost function under homogeneous input prices

The cost function as the solution to the minimization problem $\min(w_K(K)\cdot K + w_L(L)\cdot L)$ subject to $f(K,L)=q$ display the next particular form

$$C = C\big(q, w_K(1), w_L(1)\big) = q^{\frac{h+1}{r}} w_K(1)^{\frac{\alpha}{r}} w_L(1)^{\frac{\beta}{r}} B$$

with

$$B = \Big(\frac{1}{A}\Big)^{\frac{h+1}{r}} \Big[\Big(\frac{\alpha}{\beta}\Big)^{\frac{\beta}{r}} + \Big(\frac{\beta}{\alpha}\Big)^{\frac{\alpha}{r}}\Big]$$

Proof

The Lagrangean function is

$$L(K,L,\lambda) = w_K(K)\cdot K + w_L(L)\cdot L + \lambda\big(q - f(K,L)\big) =$$

$$= K^{h+1} w_K(1) + L^{h+1} w_L(1) + \lambda\big(q - f(K,L)\big)$$

The partial derivatives with respect to K, L and λ are as follows:

$$L'_K = (k+1)K^h w_K(1) - \lambda f'_K(K,L)$$

$$L'_L = (k+1)L^h w_L(1) - \lambda f'_L(K,L)$$

$$L'_\lambda = q - f(K,L) = q - f\Big(K\cdot 1, K\cdot \frac{L}{K}\Big)$$

Appendix 2

The production function $f(z_1,z_2) = z_1^2\left(1+\frac{1}{z_2}\right)$ has increasing returns to scale.

Justification : $f(tz_1,tz_2) = t^2 z_1^2\left(1+\frac{1}{tz_2}\right) = t^2 z_1^2 + t z_1^2 \frac{1}{z_2} > t z_1^2 + t\frac{1}{z_2} = t z_1^2\left(1+\frac{1}{tz_2}\right) = t\, f(z_1 z_2)$

for every $t>1$.

1. The production function $f(z_1,z_2) = z_1^2\left(1+\frac{1}{z_2}\right)$ is quasi-convex.

Justification: $H_f(z_1,z_2) = \begin{bmatrix} f''_{z_1 z_1} & f''_{z_1 z_2} \\ f''_{z_2 z_1} & f''_{z_2 z_2} \end{bmatrix} = \begin{bmatrix} 2\left(1+\frac{1}{z_2}\right) & -2\frac{z_1}{z_2^2} \\ -2\frac{z_1}{z_2^2} & 2\frac{z_1^2}{z_2^3} \end{bmatrix}$

and $\Delta_1 = 2\left(1+\frac{1}{z_2}\right) > 0$, $\Delta_2 = 4\frac{z_1^2}{z_2^3}\left(1+\frac{1}{z_2}\right) - 4\frac{z_1^2}{z_2^4} = 4\frac{z_1^2}{z_2^3} > 0$

2. The solution of the FOC in (3.21) and (3.22) is the global optimum to the constrained optimization problem in (3.2).

   Justification: The hessian matrix for the Lagrangean function in (3.4) is undefined for particular values of z_1^{opt} , z_2^{opt} and λ^{opt} in (3.21)-(3.22). We implicitely assume that the $w_1(1), w_2(1)$ and $q$are so that the denominator is non-zero and the solutions above are well defined.

References

Blinder, Alan S., 1982. Inventories and Sticky Prices: More on the Microfoundations of Macroeconomics. *American Economic Review,* 72(3), pp. 334-348.

Cowen, T. and Alex Tabarrok, 2010. *Modern Principles: Microeconomics*. Worth Publishers.

Cowell, F., 2005. *Microeconomics:Principles and Analysis*, Oxford University Press.

Dobrescu, E., 2006. *Macromodels of the Romanian Economy*. Bucuresti: Editura Economică.

Dobrescu, E., 2009. Estimating the Total factor Productivity in Romanian Economy. *Amfiteatrul Economic, 11(26), pp.512-522.*

Dobrescu, E., 2013. Restatement of the I-O coefficient stability problem. *Journal of Economic Structures,* 2(2), pp.1-84. Doi:10.1186/2193-2409-2-2.

Erdeml, H., 2009. *The Paradox of the Synonymy of Increasing Returns and Economies of Scale*. Cilt/Volume II Say/ Number 1 Nisan Soysal Bilimler.

Gordon, Robert J., 1981. Output Fluctuations and Gradual Price Adjustment. *Journal of Economic Literature,* 19, pp. 493-530.

Hall, Robert E., 1980. Employment Fluctuations and Wage Rigidity. *Brookings Papers on Economic Activity*, 10(1), pp. 91-124.

Mass-Colell, A., Whinston, M., Green, J.R., 1995. *Microeconomic Theory*, Oxford University Press.

Mortensen, Dale T., 1970. Job Search, the Duration of Unemployment, and the Phillips Curve. *American Economic Review*, 60(5), pp. 847-62.

Phelps, E., 1968. Money-Wage Dynamics and Labor-Market Equilibrium. *Journal of Political Economy*, 76(4), pp. 678-71l.