9. ON THE SOLUTIONS TO THE RAMSEY MODEL WITH LOGISTIC POPULATION GROWTH VIA THE PARTIAL HAMILTONIAN APPROACH[1]

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Abstract

In two recent papers Naz *et al*. developed a new methodology for solving the dynamic system of first-order ordinary differential equations arising from optimal control problems, by using the partial Hamiltonian approach. They derived closed-form solutions for the Ramsey model and for the Lucas-Uzawa model. As it is well-known, on the two models there are some papers which have found closed-form solutions like those of Smith, Ragni *et al.*, Viasu, Bucekkine and Ruiz-Tamarit, Hiraguchi and Chilarescu.

We knew that with no restrictions on the parameter the solution of the model of Ramsey could not be found. Almost all papers published until now tried to describe the trajectories of the variables along the steady state (equilibrium). Of course, this is not enough to understand the evolution of an economy. The most important thing is to be able to describe the trajectories from the starting point to the steady state.

We apply this new technique developed by Naz *et al*. in order to examine the existence of analytical or closed-form solutions for the Ramsey model with logistic population growth, a model recently studied by Guerrini.

Keywords: Canonical Hamiltonian; first integral; Ramsey model
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1. Introduction

In two recent published papers, Naz *et al*. (2014, 2016) studied the solutions of two endogenous growth models by using the method of the partial Hamiltonian operator. The

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method is not new and has been also used to study some economic growth models by Usher (1994), De Leon and De Diego (1998) and Askenazy (2003). As it is well known, on the two endogenous growth models there are some papers which have found closed-form solutions like those suggested by Smith (2006), Ragni *et al.* (2010), Viasu (2014), Bucekkine and Ruiz-Tamarit (2008), Hiraguchi (2009) and Chilarescu (2011).

The implications of studying the Ramsey model within a more general framework were also studied by some authors. Accinelli and Brida (2007), Guerrini (2009) and (2010) have explored the case where the change over time of the labor force is governed by the logistic growth law. Following an idea of Smith, Guerrini derives a closed-form solution for the case where the capital's share is equal to the reciprocal of the intertemporal elasticity of substitution.

The main aim of our paper is to show that the results derived by Guerrini in the above cited papers are at least questionable. Using the partial Hamiltonian approach we try to prove that an analytical or a closed-form solution for the Ramsey model under logistic growth rate does not exist. We obtain some other interesting results concerning the solutions of the Ramsey model under some particular parameterization.

The rest of the paper is structured as follows: in the second section we present some preliminary mathematical results, necessary for understanding the procedure to obtain the first integral; the third section is the main contribution of our paper and there we apply this new technique in order to obtain admissible solutions for the modified Ramsey model. In the final section we present some conclusions.

2. Some Preliminary Mathematical Results

We present in this section some necessary results for the Hamiltonian approaches to conservation laws in order to obtain the first integrals. For more details see the papers by Dorodnitsyn and Kozlov (2010) and Naz *et al.* (2014). The canonical Hamiltonian equations are:

$$\dot{x}_i = \frac{\partial H}{\partial u_i}, \quad \dot{u}_i = \frac{\partial H}{\partial x_i}, \quad i = 1,2,\dots,n, \qquad (1)$$

where: x_i, i=1, 2, …, n are the state variables and u_i, i = 1, 2, …, n are the co-state or the dual variables. Lie point symmetries in the space (t, x, u), are generated by operators of the form

$$X = \xi(t,x,u)\frac{\partial}{\partial t} + \eta^i(t,x,u)\frac{\partial}{\partial x_i} + \varsigma^i(t,x,u)\frac{\partial}{\partial u_i}, \qquad (2)$$

where, for example, x^i means

$$x^i = \sum_{i=1}^{n} x_i.$$

Definition 1 Let us now consider the Hamiltonian elementary action:

$$u_i dx^i - Hdt \qquad (3)$$

We call a Hamiltonian function invariant with respect to a symmetry operator (2) if the elementary action (3) is an invariant of the group generated by this operator.

Theorem 1 A Hamiltonian is invariant with respect to a group generated by the operator (2) if, and only if, the following condition holds

$$\varsigma^i \frac{\partial H}{\partial u_i} + u_i D(\eta^i) + X(H) - HD(\xi) = 0, \qquad (4)$$

where: D is the operator of total differentiation with respect to time

$$D = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x_i} + \dot{u}^i \frac{\partial}{\partial u_i}.$$

By using the Noether-type theorem, we can relate conservation properties of the canonical Hamiltonian equations to the invariance of the Hamiltonian function.

Lemma 1 The identity

$$\varsigma^i \dot{x}_i + u_i D(\eta^i) - X(H) - HD(\xi) \equiv \xi \left(D(H) - \frac{\partial H}{\partial t} \right) - \eta^i \left(\dot{u}_i + \frac{\partial H}{\partial x_i} \right)$$

$$+ \varsigma^i \left(\dot{x}_i - \frac{\partial H}{\partial u_i} \right) + D[u_i \eta^i - \xi H] \qquad (5)$$

is true for any smooth function H = H (t, x, u).

We call this identity the Hamiltonian identity. This identity produces the following result.

Theorem 2 The canonical Hamiltonian equations (1) possess a first integral of the form

$$I = u_i \eta^i - \xi H \qquad (6)$$

If, and only if, the Hamiltonian function is invariant w.r.t. the operator (2) on the solutions of (1).

Remark 1 Theorem 2 can be generalized to the case of the divergence invariance of the Hamiltonian action

$$\varsigma^i \frac{\partial H}{\partial u_i} + u_i D(\eta^i) - X(H) - HD(\xi) = D(V) \qquad (7)$$

where: V = V (t, x, u). If this condition holds on the solutions of the canonical Hamiltonian equations (1), then there is a first integral

$$I = u_i \eta^i - \xi H - V \qquad (8)$$

The applications of the optimal control to some theoretical problems in economics generate a system of equations of the following form:

$$\dot{x_i} = \frac{\partial H}{\partial u_i}, \quad \dot{u_i} = \frac{\partial H}{\partial x_i} + \Gamma_i, \quad i = 1,2,\dots,n, \qquad (9)$$

where: Γ_i is a non-zero function which in general is a function of t, x and u. Since the system of equations (9) is not in the canonical form (1), the Remark 1 is not applicable and therefore we need to use the new concept of partial Hamiltonian introduced by Naz *et al*.

The operator X defined in (2) is a generator of point symmetry of the current value Hamiltonian described by the system of equations (9) if:

$$\dot{\eta}^i - \dot{x}^i - X\left(\frac{\partial H}{\partial u_i}\right) = 0, \qquad \dot{\zeta}^i - \dot{u}^i X\left(\frac{\partial H}{\partial x_i} - \Gamma_i\right) = 0, \qquad i = 1,2,\dots,n \quad (10)$$

on the system of equations (9). Now, following Naz *et al*. we can introduce the concept of partial Hamiltonian operator.

Definition 2 The operator X defined in (2) is a partial Hamiltonian operator corresponding to a current value Hamiltonian described by the system of equations (9), if there is a function V (x,u,t) such that the identity

$$\varsigma^i \frac{\partial H}{\partial u_i} + u^i D(\eta^i) - X(H) - HD(\xi) + \left(\eta^i - \xi\frac{\partial H}{\partial u_i}\right)(\Gamma_i) = D(V) \qquad (11)$$

holds on the solutions of the system of equations (16) and (17). The first integral is given by (8).

Using these preliminary results, in the next section we try to find the solutions for the Ramsey model under the hypothesis that the labor force is governed by the logistic growth law.

3. Solutions for the Logistic Ramsey Model

We start this section by considering the Ramsey growth model introduced by Accinelli and Brida (2007), and later also studied by Guerrini (2010), where the economy system may be seen as a closed economy inhabited by many identical agents. The model is characterized by the well-known optimization problem.

Definition 3 The set of paths {k(t),c(t)} is called an optimal solution if it solves the following optimization problem:

$$V_0 = \max_{c(t)} \int_0^\infty \frac{[c(t)]^{1-\sigma} - 1}{1-\sigma} e^{-\rho t} dt, \qquad (12)$$

subject to

$$\begin{cases} \dot{k}(t) = [k(t)]^\beta - [\delta + \dot{a} - bL(t)]k(t) - c(t), \\ k_0 = k(0) > 0, \quad L_0 = L(0) > 0. \end{cases} \qquad (13)$$

where: β is the elasticity of output with respect to physical capital, γ represents an efficiency parameter, δ is the rate of depreciation, ρ is a positive discount factor, σ^{-1} represents the constant elasticity of intertemporal substitution and throughout this paper we suppose that σ≠β. k(t) is the physical capital and c(t) is the real consumption. Population L(t) evolves according to the differential equation

$$\dot{L}(t) = [a - bL(t)]L(t), \quad a > 0, \quad b > 0. \qquad (14)$$

This is called the Verhulst equation, and the underlying population model is known as the logistic model. Without loss of generality, we may assume that $L_0 = 1$ and thus it is easily to show that the solution is given by

$$L(t) = \frac{ae^{at}}{a - b + be^{at}}\ , with\ \lim_{t\to\infty} L(t) = \frac{a}{b} \qquad (15)$$

In this model c(t) is the control variable and k(t) is the state variable. The equations (13) give the resources constraints and initial value $k_0 = k(0) > 0$ for the state variables k(t). Of course, the state variable and the control variable are both functions of time, but when no confusion occurs, we simply write k and c. To solve the problem (12) subject to (13), we define the Hamiltonian function:

$$H = \frac{c^{1-\sigma} - 1}{1 - \sigma} + \left[k^\beta - (\delta + a - bL)k - c\right]\lambda.$$

The boundary conditions include the initial value k_0 and the transversality condition:

$$\lim_{t\to\infty} k(t)\lambda(t)e^{-\rho t} = 0.$$

In an optimal program the control variable is chosen so as to maximize H. We notice that along the optimal path, λ is a function of t only. The necessary first order conditions for c to be an optimal control are:

$$\dot{k} = k^\beta - (\delta + a - bL)k - c = \frac{\partial H}{\partial \lambda}, \qquad (16)$$

and

$$\dot{\lambda} = -\lambda[\beta k^{\beta-1} - \delta - a + bL] + \rho\lambda = -\frac{\partial H}{\partial k} + \Gamma, \qquad (17)$$

and knowing that $\lambda = c^{-\sigma}$, we immediately obtain:

$$\dot{c} = \sigma^{-1}\left[\beta k^{\beta-1} - \delta - a + bL - \rho\right]c. \qquad (18)$$

These equations, together with the two boundary conditions, initial condition and transversality condition, constitute the dynamic system which drives the economy over time. Without loss of generality, we suppose that

$$X = \xi(t,k)\frac{\partial}{\partial t} + \eta(t,k)\frac{\partial}{\partial k}, \qquad (19)$$

that is the functions ξ and η depend only In the state variable k and do not depend on the co-state variable λ and thus, the identity (11) becomes

$$c^{-\sigma}\{\eta_t + \eta_{k\eta}[k^\beta - (\delta + a - bL)k - c]\} - \eta(\beta k^{\beta-1} - \delta - a + bL)c^{-\sigma}$$
$$-\left\{\frac{\sigma c^{1-\sigma}}{1-\sigma} + [k^\beta - (\delta + a - bL)k]c^{-\sigma}\right\}\{\xi_t + \xi_k[k^\beta - (\delta + a - bL)k - c]\}$$
$$+ \rho c^{-\sigma}[\eta - \xi(k^\beta - (\delta + a - bL)k - c)]$$
$$= V_t + V_k[k^\beta - (\delta + a - bL)k - c]. \qquad (20)$$

Separating Eq.(20) with respect to powers of the control variable c, we get

$$c^{2-\sigma}: \xi_t = 0. \qquad (21)$$

$$c^{1-\sigma}: \eta_k = \rho\xi - \frac{\sigma}{1-\sigma}\xi_t \qquad (22)$$

$$c^{-\sigma}: [k^\beta - (\delta + a - bL)k](\eta_k - \rho\xi - \xi_t) + \eta_t$$
$$- \eta(\beta k^{\beta-1} - \delta - \rho - a + bL) = 0. \qquad (23)$$

$$c^1:\ V_k = 0. \qquad (24)$$

$$c^0:\ V_t = \frac{1}{1-\sigma}\xi_t. \qquad (25)$$

From Eq. (21) we conclude that $\xi = \xi(t)$ and Eq.(22) enables us to write

$$\eta(t,k) = \left(\rho\xi - \frac{\sigma}{1-\sigma}\dot{\xi}\right)k \qquad (26)$$

and

$$\eta_k = \rho\xi - \frac{\sigma}{1-\sigma}\xi, \quad \eta_t = \left(\rho\dot{\xi} - \frac{\dot{\sigma}}{1-\sigma}\ddot{\xi}\right)k.$$

Introducing this result into Eq. (23) and separating with respect to powers of the state variable k yields

$$-k^\beta\left[\frac{1-\beta\sigma}{1-\sigma}\dot{\xi} + \beta\rho\xi\right] + k\frac{1}{1-\sigma}\{[\rho(1-2\sigma) + (1-\sigma)A]\dot{\xi} - \sigma\ddot{\xi} + \rho(1-\sigma)(\rho + A)\xi\} = 0, \qquad (27)$$

with $A = \delta + a - bL$. This identity if the two following differential equations have a common non-zero solution:

$$\dot{\xi} = -\frac{\beta\rho(1-\sigma)}{1-\beta\sigma}\xi, \qquad (28)$$

and

$$\sigma\ddot{\xi} - [\rho(1-2\sigma) + (1-\sigma)A]\dot{\xi} - \rho(1-\sigma)(\rho + A)\xi = 0. \qquad (29)$$

Obviously, equation (28) is valid if βσ≠1, for the case where the capital's share is not equal to the intertemporal elasticity of substitution. Now, it is clear that an analytical or a closed-form solution does not exist for the general case, with no restrictions on the parameters and we can claim that the result obtained by Guerrini is at least questionable, even if we suppose that σ = β.

Let us now consider the case a ≠ 0 and b ≠ 0. For the equation (29) we must have

$$\Delta = [\rho(1-2\sigma) + (1-\sigma)A]^2 + 4\sigma\rho(1-\sigma)(\rho + A) = 0$$

from which we obtain $A = -\frac{\rho}{1-\sigma}$ and $L = \frac{(\delta+a)(1-\sigma)+\rho}{b(1-\sigma)}$

which means that the population is constant. Under this assumption it immediately follows that β = 1 and we are in the case of the Ak model.

To obtain analytical solutions, two alternatives are to be considered. In the first case we may choose a = b and, consequently, we get L = 1 and, thus, we obtain the solution determined by by Barro and Sala-i-Martin and Ragni *et al*. If we choose b = 0, that is $L(t) = e^{at}$, we can find another admissible solution. Under this hypothesis we have

$$\frac{\rho(1-\beta)}{1-\beta\sigma} = \frac{\rho + (1-\sigma)(\delta + a)}{\sigma} \Rightarrow \sigma = \frac{\delta + \rho + a}{\beta(\delta + a)}.$$

Thus, there is also an admissible common solution of equations (28) and (29) and this solution is given by

$$\xi(t) = \pi e^{-(\delta+\alpha)(1-\beta)t}, \qquad (30)$$

where π is an arbitrary positive constant. Introducing this result into Eqs. (25) and (26) we finally get

$$V(t) = -\frac{\beta\pi(\delta +)}{\rho + (\delta + a)(1-\beta)} e^{-[\rho+(\delta+\alpha)(1-\beta)]t} \qquad (31)$$

$$\eta(t,k) = -\pi(\delta + a)ke^{-(\delta+a)(1-\beta)t}. \qquad (32)$$

According to Eq. (8), via the first integral we can thus write

$$k^{\beta} = \frac{\rho + \delta + a}{\rho + (\delta + a)(1-\beta)}c - \frac{I}{\pi}e^{[\rho+(\delta+\alpha)(1-\beta)]t}c^{\sigma}. \qquad (33)$$

As it is well-known, the steady-state values for the variables k and c are constants, given by:

$$k_*^{1-\beta} = \frac{\beta}{\rho + \delta + a}, \quad c_* = \frac{\rho + (\delta + a)(1-\beta)}{\beta} k_* \qquad (34)$$

and, therefore, the only admissible value for I is zero. Under this hypothesis, the Eq. (33) yields

$$c = \frac{\rho + (\delta + a)(1-\beta)}{\rho + \delta + a} k^{\beta} \qquad (35)$$

and after replacement into Eq. (16) we come to the following differential equation

$$\dot{k} = -(\delta + a)k + \frac{\beta(\delta + a)}{\rho + \delta + a} k^{\beta}, \qquad (36)$$

the solution of which is given by

$$k(t) = \left[\left(k_0^{1-\beta} - \frac{\beta}{\rho + \delta + a}\right) e^{-(\delta+a)(1-\beta)t} + \frac{\beta}{\rho + \delta + a}\right]^{\frac{1}{1-\beta}}, \qquad (37)$$

and this solution coincides exactly with the solution determined by Barro and Sala-i-Martin and Ragni *et al.* if we choose a = 0. In fact, we can conclude from Eq. (35) that this solution was obtained under the assumption of the constancy of the saving rate and therefore the growth rates of consumption and physical capital are not the same, more precisely $g_c = \beta g_k$, that is $g_c < g_k$.

4. Conclusions

The aim of this paper is first of all a theoretical one. We tried to discuss and to clarify the existence of analytical solutions or of closed-form solutions to the Ramsey model under the restriction of a logistic trajectory for the variable L. This model was recently studied by Guerrini and he claimed that a closed-form solution exists. We prove in this paper that such a claim is at least questionable, even for the particular case examined by Smith. To do this we use the method of partial Hamiltonian approach recently developed by Naz *et al*. We also prove that all other solutions derived by Barro and Sala-i-Martin, and by Ragni *et al.* can be easily obtained by the method described in this paper.

As it is well-known, with no restrictions on the parameter, the solution of the Ramsey model could not be found. Almost all papers published until now tried to describe the trajectories of the variables along the steady state (equilibrium). The most important thing is to describe the trajectories from the starting point to the steady-state. Unfortunately, for the case of this model (and not only this model), the numerical methods are not conceivable, simply because the starting value of the control variable is not known and it could not be determined.

The differential equation describing the trajectory of L has an analytical solution and that is why we may consider this variable as an exogenous vari- able. Under this hypothesis, the variable L doesn't influence the Hamiltonian function. This hypothesis is, of course, one of the limits of the model studied in this paper and in the cited papers. In a future paper, we intend to present the more general case, where the variable L is an endogenous variable, and consequently, with effect on the Hamiltonian function.

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